ON THE USE OF SUMS OF SINES IN THE DESIGN OF SIGNAL WINDOWS

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ABSTRACT

Windowing of discrete signals by temporal weighting is an essential tool for spectral analysis and processing to reduce bias effects. Many popular weighting functions (e. g. Hann, Hamming, Blackman) are based on a sum of scaled cosines. This paper presents an alternative class of windows, constructed using sums of sines and exhibiting unique spectral behavior with regard to zero location and a side lobe decay of at least –12 dB/octave due to guaranteed continuity of the weighting. The parameters for the 2- and 3-term realizations with minimum peak side lobe level are provided. Usage of the sum-of-sines windows with the DFT and their adoption to lapped transforms such as the MDCT are also examined.

1. INTRODUCTION

Finite impulse response (FIR) filtering of discrete signals, particularly in the context of filter banks, is widely utilized in spectral analysis, processing, synthesis, and media data compression, amongst other applications. It is well understood that the temporal (or spatial) finiteness of the filter(s), and hence the finiteness of the signal interval which can be processed at a time, can lead to a characteristic referred to as bias or leakage [1, 2, 3]. The cause of bias can be ascribed to time-frequency uncertainty or, put differently, discontinuities between the edges of the interval's waveform as well as those of its differentials. To reduce the unwanted effects related to spectral leakage, it is therefore often necessary to minimize such discontinuities in the signal and some of its differentials. This can be accomplished by multiplying each sample $s(t)$, $t = 0, 1, \ldots, L-1$, of the L-length interval by a weight $w(t)$ prior to filtering, such that the endpoints of the waveform are tapered to zero. An equivalent approach is to apply the weights to each basis filter of the filter bank [2].

Since the weighting factors are often described by an analytical expression, a set of factors is commonly known as a weighting function or window function. A multitude of window functions, optimized toward different criteria, have been documented [1, 2, 3, 4, 5]. Arguably three of the most popular functions in use today are the ones reported by von Hann, Hamming, and Blackman. This paper proposes alternatives to these functions, equally easy to compute and with unique spectral performances in terms of bias reduction.

The remainder of the document is organized as follows. Section 2 revisits the aforementioned window functions and identifies the underlying general design equation. Section 3 then presents a modification of this expression to define an alternative class of windows. In Section 4, the performance of 2- and 3-term variants of this window class is evaluated and compared to other windows using some of the figures of merit described in [2]. Motivated by the result, specially optimized realizations are derived. Section 5 reports on an interesting feature of the proposed window class when used with the DFT, and Section 6 studies the feasibility of applying the previously derived approach to the design of power complementary window functions for use with block-based transforms like the MDCT. Section 7 concludes the paper.

2. SOME CLASSIC WINDOW FUNCTIONS

For the sake of consistency and comparability with seminal investigations of window functions, Nuttall's methodology and notation [4] shall be adopted in the present discussion. In particular, let L denote the duration (length) of a window realization, t the location (time) within the weighting, and f the frequency within the window's power density spectrum, obtained by Fourier transformation of the window function. Moreover, all window functions shall be normalized to peak amplitude of one. Since only symmetrical, even-length, bell-shaped windows are studied here, this implies $w(L/2) = 1$.

The first weighting function to be considered is known as the Hann (or Hanning) function. It is specified in [2] as

$$w_{Hann}(t) = \sin^2\left(\pi \cdot \frac{t}{L}\right) \tag{1}$$

for DSP applications (nonnegative values of t). As shown in [2] and evident from (1), the Hann function is a special case of a class of exponentiated sine functions:

$$w_a(t) = \sin^a\left(\pi \cdot \frac{t}{L}\right),\ a \geq 0\ . \tag{2}$$

Note that (1) can also be written as the sum of an offset and a scaled cosine:

$$w_{Hann}(t) = 0.5 - 0.5\cos\left(2\pi \cdot \frac{t}{L}\right)\ . \tag{3}$$

This formulation allows for a particular spectral optimization of the Hann window (see Section 4) by changing the offset and scaling factor [2]. The outcome is the Hamming function, whose exact parameterization is given in [4] as

$$w_{Hamming}(t) = 0.53836 - 0.46164\cos\left(2\pi\cdot\frac{t}{L}\right). \quad (4)$$

As pointed out by Nuttall [4], the Hann and Hamming windows are two-term realizations of a class of (K+1)-term functions which shall be referred to as *sum-of-cosines* functions. Simplifying Nuttall's notation, they can be written as[1]

$$w_b(t) = \sum_{k=0}^{K}(-1)^k b_k \cos\left(2k\pi\cdot\frac{t}{L}\right) \quad (5)$$

for usage in DSP applications. Three-term implementations are also common. A simple case is (5) with $K=2$ and factors

$$b_0 = 0.375,\ b_1 = 0.5,\ b_2 = 0.125\ , \quad (6)$$

which is equivalent to (2) with $a = 4$. Similar to Hamming's approach, Blackman [1] derived the following optimized b_k:

$$b_0 = 0.42,\ b_1 = 0.5,\ b_2 = 0.08\ . \quad (7)$$

Nuttall [4] further refined Blackman's values for better near-field spectral response (first side lobes, see Section 4):

$$b_0 = 0.40897,\ b_1 = 0.5,\ b_2 = 0.09103\ . \quad (8)$$

The interested reader is encouraged to take a look at [4] for other optimized 3- and 4-term sum-of-cosines windows.

3. THE SUM-OF-SINES CLASS OF WINDOWS

In the preceding section, it was noted that equation (2) with $a = 2$, that is, $w_2(t)$, is equivalent to (5) with $K = 1$, $b_0 = 0.5$, $b_1 = 0.5$. Moreover, equivalence between $w_4(t)$ and (5) with $K = 2$ and b_k of (6) was established. The question now arises as to which b_k yield $w_1(t)$, $w_3(t)$, or more generally, any $w_a(t)$ with odd a. Observing (2) and (5), it becomes clear that it is impossible to construct a sum-of-cosines window which is equivalent to an odd-exponentiated sine window. However, in some applications where odd-a $w_a(t)$ are required, it may be desirable to use a formulation similar to (5) to allow for spectral leakage optimizations as carried out by Hamming, Blackman, and Nuttall. Luckily, the *sum-of-sines* functions

$$w_c(t) = \sum_{k=0}^{K}(-1)^k c_k \sin\left((2k+1)\pi\cdot\frac{t}{L}\right) \quad (9)$$

provide the necessary means for optimization. By choosing the constants c_k suitably, two features can be acquired.

First, a window corresponding to an odd-exponentiated sine window of (2) can be constructed. The c_k for the three lowest-order odd-a $w_a(t)$ shall be specified here. The classic sine window $w_1(t)$ is trivial to construct using (9) by setting $K = 0$ and $c_0 = 1$. For $w_3(t)$, K is increased to $K = 1$, and

$$c_0 = 0.75,\ c_1 = 0.25\ . \quad (10)$$

The fifth-order $w_5(t)$ is finally obtained using $K = 2$ and

$$c_0 = 0.625,\ c_1 = 0.3125,\ c_2 = 0.0625\ . \quad (11)$$

Second, like the b_k in (5), the c_k can be determined such that spectral behavior similar to that of the Blackman, Hamming, and Nuttall windows is achieved. Before deriving the respective c_k for $K = 1$ and $K = 2$, though, it is important to assess exactly which aspect of a window's spectral response should be optimized. To this end, objective measures of the spectral performance of a window are necessary. In the next section, an analysis of all window functions mentioned thus far is conducted by means of some popular measures.

4. EVALUATION AND OPTIMIZATION

It is well established that the multiplication of a time signal by another signal corresponds to the convolution of the frequency transforms of the two signals. Hence, by applying a weighting function to a signal, the signal's spectrum is convolved with the spectrum of the weighting. To evaluate the effect of a window function, it therefore suffices to study its spectrum, for instance using Fourier transformation.

Figures 1 and 2 illustrate the magnitudes of the power spectra of the above windows, normalized in frequency and amplitude as in [4]. Due to recurring spectral zeros, all windows exhibit a main lobe at zero frequency and side lobes decaying in amplitude with increasing frequency. The falloff rate of the side lobes is dictated by the discontinuities at the edges of the window function as well as those of its differentials; the more low-order derivatives are continuous, the faster a window decays to zero for large f. See also [2, 4].

For the exponentiated sine functions $w_a(t)$ of Figure 1, it can be stated that the asymptotic falloff in dB per octave is proportional to a [6]:

$$\text{falloff}(w_a) = -6.02(a+1)\ \frac{\text{dB}}{\text{oct}}\ . \quad (12)$$

This appears to hold for all nonnegative real a, not only integers. For the optimized windows of Figure 2, a different side lobe behavior can be observed. The Hamming window, whose main lobe width equals that of $w_2(t) = w_{Hann}(t)$, falls off at only –6 dB per octave because the weighting function is not continuous. Similarly, the Blackman and Nuttall windows, which have the same main lobe width as $w_4(t)$, show a decay of only –18 dB per octave; their first derivatives of weighting are continuous, but their third derivatives are not. However, these windows exhibit lower *maximum* side lobe levels than their $w_a(t)$ counterparts. This can lead to notably

1 This equals [4, eqn. (11)] with the leading scalar $1/L$ omitted.

reduced spectral bias in some applications and is the reason why the optimized windows were developed.

Since the optimization procedure used for the sum-of-cosines windows in Figure 2 can also be applied to the sum-of-sines functions of (9), it is possible to modify the 2-term window with (10) and the 3-term window with (11) for the lowest maximum side lobe level (the one-term sine window with $c_0 = 1$ cannot be optimized this way). Due to the use of sinusoids, any realization of (9) approaches zero amplitude at its endpoints; a side lobe falloff rate of –12 dB per octave ($1/f^2$, see [2]) is therefore guaranteed. If the derivatives are allowed to be discontinuous, additional degrees of freedom are obtained for determining the c_k, which can be employed to minimize the peak side lobe magnitude [4].

For the two-term sum-of-sines window ($K = 1$), the admission of a discontinuous first derivative yields one extra degree of freedom in the choice of c_0 and c_1. It is found that

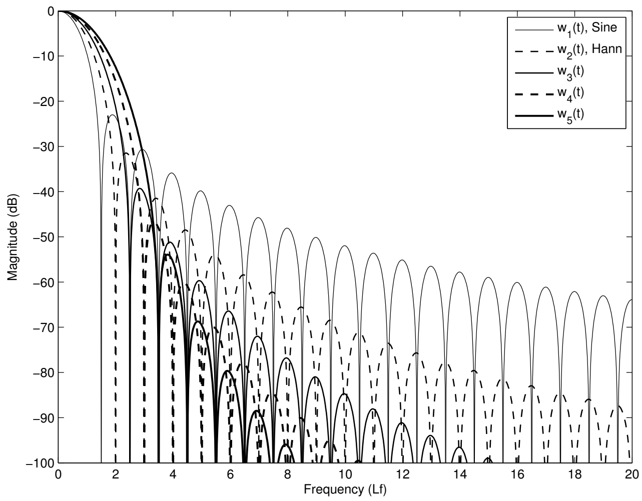

Figure 1: *Spectra of some exponentiated sine functions (2).*

$$c_0 = 0.79445,\ c_1 = 0.20555 \tag{13}$$

produce the lowest possible side lobe maximum of –54.3 dB (first and third side lobe). The 3-term window ($K = 2$) offers two extra degrees of freedom in the selection of the c_k. The minimum peak side lobe level of –82.8 dB is reached using

$$c_0 = 0.69295,\ c_1 = 0.2758,\ c_2 = 0.03125\ . \tag{14}$$

Figure 3 shows the power spectra of windows (13) and (14). For all ten presented windows, the maximum side lobe level, the asymptotic falloff, the main lobe width (as given by the location of the first zero), and the 6-dB bandwidth (a measure of the resolution of a window, see [2]) are listed in Table 1. Note how in terms of overall spectral performance, window (13) lies right between the 2-term Hamming and 3-term Nuttall window. Moreover, while achieving a side lobe peak similar to that of the Blackman window, window (13) has a narrower main lobe. Window (14) has the lowest side lobe maximum of all windows in this discussion, but along with $w_5(t)$, it also exhibits the widest main lobe.

5. SUM-OF-SINES WINDOWS AND THE DFT

The observant reader will have noticed the difference in the zero locations between the spectra of the sum-of-sines and the sum-of-cosines windows. As apparent in the figures, for the latter windows, most or all zeros occur at integer multiples of Lf, whereas for the sum-of-sines windows, the zeros lie halfway between integer Lf. In the following, this feature shall be illuminated with regard to analyzing the spectra of windowed harmonic signals using the DFT.

As noted earlier, the Fourier transform (FT) of a signal interval $s(t)$ weighted by $w(t)$ is equivalent to the convolution of the individual FTs of $s(t)$ and $w(t)$. The FTs of the sine window $w_1(t)$ and the Hann window $w_2(t)$ are given by

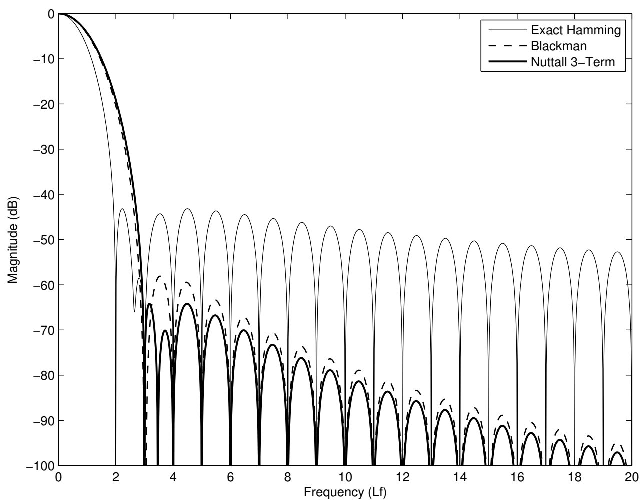

Figure 2: *Spectra of optimized sum-of-cosines functions (5).*

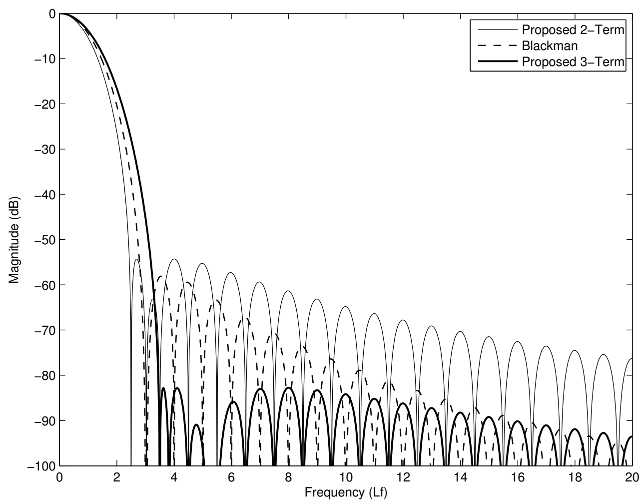

Figure 3: *The proposed optimized sum-of-sines windows (9).*

Table 1: *Figures of merit for the presented windows.*

Window Function	Side Lobe Maximum (dB)	Side Lobe Decay (dB/oct.)	Main Lobe Width (Lf)	6-dB Bandwidth (Lf)
$w_1(t)$, 1-term sine	–23.0	–12	3	1.64
$w_2(t)$, Hann	–31.5	–18	4	2.00
$w_3(t)$, 2-term sine	–39.3	–24	5	2.31
$w_4(t)$, 3-term cos.	–46.7	–30	6	2.59
$w_5(t)$, 3-term sine	–53.9	–36	7	2.84
Exact Hamming	–43.2	– 6	4	1.82
Proposed 2-term	**–54.3**	**–12**	**5**	**2.10**
Blackman	–58.1	–18	6	2.30
Nuttall 3-term	–64.2	–18	6	2.36
Proposed 3-term	**–82.8**	**–12**	**7**	**2.48**

$$W_1(f) = \frac{2\cos(\pi f)}{\pi(1-4f^2)} \tag{15}$$

and

$$W_2(f) = \frac{\sin(\pi f)}{2\pi f(1-f^2)} , \tag{16}$$

respectively [3]. Thus, $W_1(f) = 0$ for $f = n + 0.5$, $|n| \geq 1$, and $W_2(f) = 0$ for $f = n$, $|n| \geq 2$, with n being an integer. The FTs of the higher-order and optimized windows of Table 1 differ from (15) and (16), but the respective trigonometric term in the numerator (cos() for the sum-of-sines, sin() for the sum-of-cosines windows) is common to all. In the context of the DFT, the implication is that maximum spectral leakage with a sum-of-cosines window coincides with minimum leakage with a sum-of-sines window, and vice versa. An example is given in Figure 4 for the proposed 2-term window (13) and Nuttall's 3-term window (8) applied in a 256-point DFT.

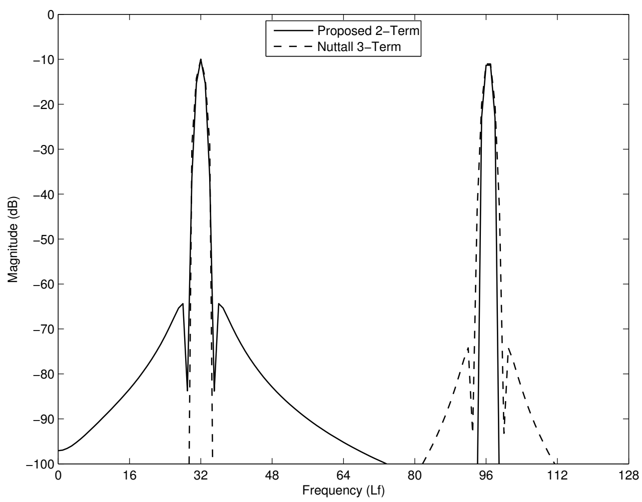

Figure 4: *DFT spectra of two sinusoids with frequencies of Lf = 32 and 96.5, after applying different window functions.*

6. SUM-OF-SINES WINDOWS AND THE MDCT

In contemporary audio or video coders, a signal waveform is divided into segments, and each segment is quantized to a coarser representation to obtain high data compression, i. e. a low bit rate required for storing or transmitting the signal. In an attempt to achieve a coding gain by means of energy compaction (or in other words, to increase perceptual quality of the coded signal for a given bit rate), filter-bank transformations of the segments prior to quantization have become popular. Most recently developed systems apply time-to-frequency transformation in the form of the modified discrete cosine transform (MDCT), a filter bank permitting adjacent segments to overlap while providing critical sampling.

For better performance, the forward and inverse MDCT operations are accompanied by weighting of each segment: on encoder side, an analysis window is employed before the MDCT, and on decoder side, a synthesis window is applied after the inverse MDCT. Unfortunately, not every weighting function is suitable for use with the MDCT. Assuming identical, symmetrical analysis and synthesis window functions,

$$w(L-1-t) = w(t), \; t = 0, 1, ..., T-1, \tag{17}$$

the entire system can only yield perfect input reconstruction in the absence of quantization or transmission errors if

$$w^2(t)+w^2(T+t) = 1, \; t = 0, 1, ..., T-1, \tag{18}$$

with $T = L/2$. This is the so-called Princen-Bradley or power complementarity (PC) condition reported in [7]. Common PC windows are the sine and KBD windows utilized in the MPEG-2/-4 AAC standard [6, 8], with the former given by

$$w_{sine}(t) = \sin\left(\pi \cdot \frac{t+0.5}{L}\right) , \tag{19}$$

as well as the window of the Vorbis codec specification [9],

$$w_{vorbis}(t) = \sin\left(\frac{\pi}{2} \cdot \sin^2\left(\pi \cdot \frac{t+0.5}{L}\right)\right) . \tag{20}$$

To investigate if equation (9) can be used to create sum-of-sines windows satisfying (18), we note that, given (17), $w_{sine}(t)$ can be regarded as the sine of a triangular function:

$$\tau(L-1-t) = \tau(t) = \frac{t+0.5}{T} , \tag{21}$$

$$w_{sine}(t) = \sin\left(\frac{\pi}{2} \cdot \tau(t)\right) . \tag{22}$$

Likewise, $w_{vorbis}(t)$ can be written as (22) with $\tau(t)$ replaced by

$$\tau'(t) = \sin^2\left(\frac{\pi}{2} \cdot \tau(t)\right) . \tag{23}$$

The amplitude complementarity about $T = L/4$ of (21) and (23),

$$\tau(t)+\tau(T-1-t) = 1,\ t = 0, 1, ..., L/4-1, \quad (24)$$

suggests that alternatives to these functions can be designed to optimize the frequency response of the window function without sacrificing the PC property. In fact, upholding (17),

$$\tau_d(t) = \tau(t)-\sum_{k=1}^{K} d_k \sin\left(2k\pi\cdot\tau(t)\right) \quad (25)$$

is an extension of (21) conforming to (24), which employs a modification of the sum-of-sines function of (9); the alternating-sign term is omitted, and instead of odd multiples of π, even multiples are considered. Informal experiments run by the present author indicate that, although PC is obtained even with d_k yielding $\tau_d(t) < 0$ for some t, only realizations with nonnegative $\tau_d(t)$ for all t yield satisfactory pass-band selectivity and stop-band rejection simultaneously.

In Section 4, the c_k coefficients of (9) were chosen such that the maximum side lobe level of the resulting window is minimized. A similar procedure can be followed here. However, owing to the PC constraint of (18), the spectral design possibilities are more limited, especially regarding the first two or three side lobes. In general, one must specify a lower frequency border $Lf_0 > 1.5$ (or alternatively, a start side lobe) above which the side lobe maximum can be minimized by a reasonable amount. To give an example, an informal exhaustive search with $Lf_0 = 4.5$ yields the 2-term parameterization

$$d_1 = 0.12241,\ d_2 = 0.00523\ , \quad (26)$$

which produces a window whose first three side lobes above Lf_0 all have a level of –66.8 dB. The higher-frequency side lobes decay from that value at a rate of –12 dB per octave, just like those of the optimized windows (13) and (14) of the previous sections. The frequency response of the weighting function constructed using (17), (22), (25) and (26) is shown in Figure 5 along with those of $w_{sine}(t)$ and $w_{vorbis}(t)$. Clearly, a substantial increase in side lobe rejection is achieved in the proposed window in comparison to the sine window. Due to constraint (18), this advantage comes at the cost of a slightly wider main lobe and higher first side lobe. A comparison to the Vorbis window shows almost identical main lobe widths and maxima of the first two side lobes. For $4.5 < Lf_0 < 11.5$, the proposed window outperforms $w_{vorbis}(t)$ in terms of side lobe attenuation. Note also that the Vorbis window spectrum falls off at –18 dB per octave and has its magnitude zeros at (or near) integer multiples of Lf. Hence, its spectral behavior resembles that of a sum-of-cosines window. In fact, it may be considered the PC equivalent of the Hann window. Likewise, the proposed PC window seems to be a counterpart of the optimized sum-of-sines windows of Section 4. A more thorough investigation, including a performance evaluation in the context of audio coding, is a topic for future research.

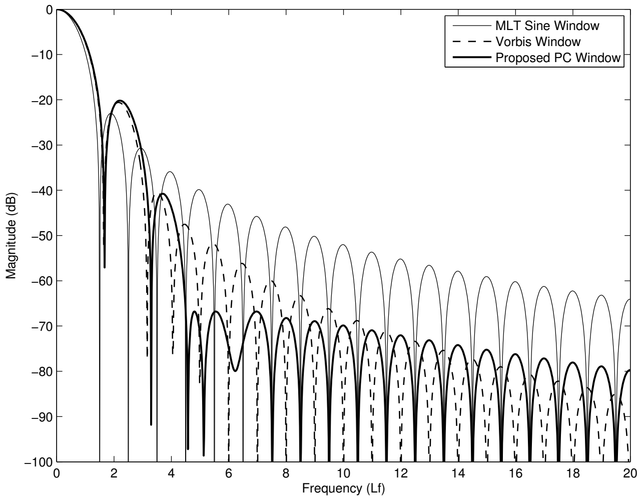

Figure 5: *Spectra of two PC windows and proposed window*

7. CONCLUSION

Mathematically simple alternatives to the Hamming, Blackman, and similar windows, generated using sums of weighted sines, have been presented. The sum-of-sines approach yields unique properties such as guaranteed continuity of the window function and can also be applied in the construction of power complementary windows for e. g. audio coding.

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