

PHYSICAL MODELING OF THE HARPSICHORD PLECTRUM-STRING INTERACTION

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ABSTRACT

In this paper, we present a thorough treatment of a harpsichord plectrum-string interaction which allows for large end deflections and both transverse motions of the string. We start from the general equations of motion of a bent beam, and an accurate shape of the plectrum is calculated, agreeing with existing known cantilever beam models when end deflections are assumed small. All the governing forces on the string are considered, and the complete motion of the string up to its release is simulated, allowing for future implementation on physical model sound synthesis of strings. Simulation results agree with what is experienced playing a real harpsichord string.

1. INTRODUCTION

The harpsichord is a plucked keyboard instrument known to exist first around late 14th to mid 15th century. It continued to be developed and flourished during the 16th and 17th centuries, becoming an important instrument both in accompaniment, more complex continuo playing, and other virtuosic solo keyboard works. Although its popularity dropped sharply after the advent of the piano, 20th century early music revival movements, also known as *historically informed performances*, have since brought the harpsichord back to the music scene.

As a member of the stringed keyboard instrument family, the harpsichord is the only instrument whose strings are plucked. (Virginals and spinets are members of the general harpsichord family.) As shown in Figure 1, this is done by having a piece of thin flexible material called the plectrum, attached firmly onto a jack, pluck the strings. The jack sits at the end of the keyboard lever, and when a key is played, the jack moves up and plucks the string. Two sets of lower and upper guides constrain the jack to move completely in the vertical direction. Historically the plectrum was made out of bird quill, but modern plectra often use plastics like Delrin.

The general physics of the harpsichord has been addressed by Kellner [1], Fletcher [2], and Spencer [3], discussing many aspects

of the design of a harpsichord. More specific studies on the harpsichord were done on the air and soundboard vibration modes [4, 5] and the attack transients of the harpsichord tone [6, 7].

One aspect of the harpsichord that has not been that well investigated, however, is the interaction between the plectrum and the string. A generally accepted distinguishing characteristic of the harpsichord is its inability to create dynamics. There are still some minor differing opinions on this issue, but despite recent studies and findings [8], the harpsichord's dynamic expressivity remains limited. A theoretical model was proposed by Griffel [9], which prompted further studies to be carried out by Giordano and Winans [10]. A more recent plectrum model can also be found in [11]. Differences in radiated sound due to changes in guitar plectrum thickness have been reported in [12]. It is clear that plectrum-string interactions are important in the sound production of the instrument. A more precise plectrum model is thus important both in understanding the physical nature of the instrument sound and also in digital sound synthesis of the instrument.

An existing physical model of the harpsichord has been implemented in [13], using a sampled excitation signal database as the input. Modal synthesis of plucks have been presented in [14] as well as the aforementioned physical model in [11], with the guitar body model in mind. Synthesized excitation allows for controllable parameters, which gives more expressivity to the sound synthesis. Unlike [11], where the string motion is constrained to move only in one transverse direction, here we present a plectrum model where the string has transverse motion in two orthogonal directions, and the end plectrum deflection is not necessarily assumed to be small.

The paper is organized as follows. In Section 2, a more general model is proposed for the plectrum-string interaction. In Section 3, simulation results are discussed for different parameters. Lastly, conclusions and ideas for further research are drawn in Section 4.

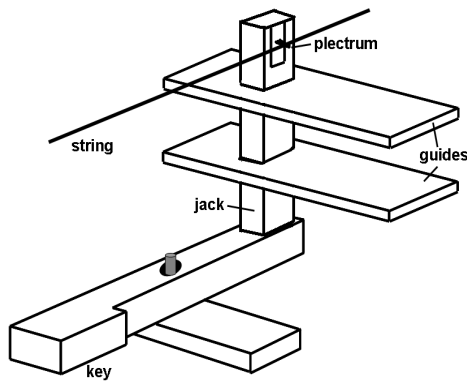


Figure 1: Harpsichord key and jack.

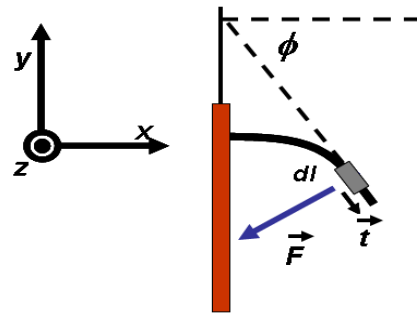


Figure 2: Force exerted on plectrum.

2. PLECTRUM-STRING INTERACTION MODEL

2.1. Model Assumptions

While treating the bending of a plectrum as a cantilever beam has been discussed before, those models assume small strain and small deflections [11]. This, in turn, also assumes that the force applied is in the vertical direction only. Observing a harpsichord string being plucked, however, one notices that the end of the plectrum undergoes significant deflection, and that the string exhibits sideways transverse motion as well as vertical. Before carrying forth with our model, we would like to make the following assumptions:

Plectrum elastic properties:

- Small strain within the plectrum (still allows for large end deflection)
- The plectrum is an isotropic elastic material
- The plectrum is modeled as a beam with uniform rectangular cross section
- Bending of the plectrum occurs only in a plane so that there is no twisting motion on the end of the plectrum

Force considerations:

- There is no friction between string and plectrum
- Force exerted on plectrum always perpendicular to plectrum surface in contact
- Force concentrated only at one point
- The plectrum and string motion assumed to be quasi static

2.2. Plectrum Model

The plectrum is treated as a thin bar clamped on one end, with length L , Young's modulus E , and second moment of inertia I . If the thin bar is subject to a force \vec{F} at the free end, and the resulting bending moment is \vec{M} from the internal stresses, the general equilibrium equation for a bent rod can be found as

$$\frac{d\vec{M}}{dl} = \vec{F} \times \vec{t} \quad (1)$$

where dl is an infinitesimal element of the rod, and \vec{t} is a unit vector tangential to the rod. Under the assumptions made in section 2.1, the bending moment is simplified to

$$\vec{M} = EI \frac{d\vec{r}}{dl} \times \frac{d^2\vec{r}}{dl^2} \quad (2)$$

where \vec{r} is the radius vector from a fixed point to the point considered on the rod. We can define a spatial coordinate where the x - y plane denotes the plane of the bent rod and z is the coordinate perpendicular to this plane. Defining ϕ as the angle between the horizontal and \vec{t} , as in Figure 2, then equation (1) can be further simplified:

$$EI \frac{d^2\phi}{dl^2} + F = 0 \quad (3)$$

A first integration gives

$$EI \frac{d\phi}{dl} + c_1 = -Fl \quad (4)$$

where integration constant c_1 can be found by imposing the boundary condition on the free end ($l = L$):

$$\frac{d\phi}{dl} = 0 \quad (5)$$

Performing a second integration, we arrive at

$$\phi(l) = \frac{F}{EI} (Ll - \frac{1}{2}l^2) \quad (6)$$

$$\phi_0 \equiv \phi(l = L) = \frac{1}{2} \frac{FL^2}{EI} \quad (7)$$

and the parametric shape of the plectrum $x(l)$ and $y(l)$ can be found:

$$\begin{cases} x(l) = \int \cos \phi dl \\ y(l) = - \int \sin \phi dl \end{cases} \quad (8)$$

For small angle approximations of $\phi \ll 1$, equation (8) reduces to the familiar results of ordinary cantilever beam load equations:

$$\begin{cases} x = l \\ y = - \frac{Fl^2}{6EI} (3L - l) \end{cases} \quad (9)$$

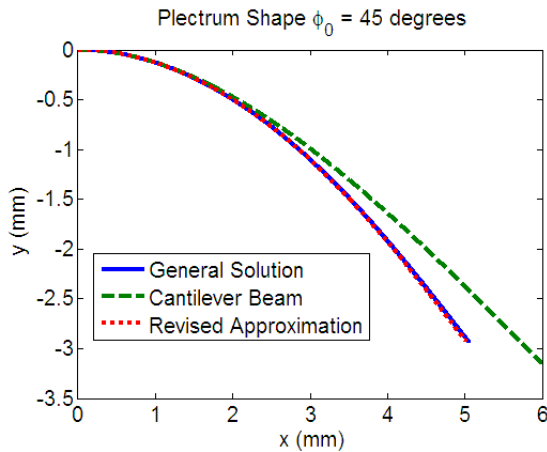


Figure 3: Bent shape models of the plectrum with a targeted end deflection angle $\phi_0 = 45^\circ$. The General Solution and Revised Approximation are nearly indistinguishable in this figure.

A closer approximation is found by including the second-order expansion terms of $\cos \phi$ and $\sin \phi$ in equation (8). The result we arrive at is

$$\begin{cases} x(l) = l - \frac{1}{2} \left(\frac{F}{EI} \right)^2 \left(\frac{L^2 l^3}{3} - \frac{Ll^4}{4} + \frac{l^5}{20} \right) \\ y(l) = -\frac{Fl^2}{6EI} (3L - l) + \frac{1}{6} \left(\frac{F}{EI} \right)^3 \left(\frac{L^3 l^4}{4} - \frac{3L^2 l^5}{10} + \frac{Ll^6}{8} - \frac{l^7}{56} \right) \end{cases} \quad (10)$$

and it gives good agreement to the general solution for end deflection angles ϕ even up to 45° , as shown in Figure 3. Our revised model follows the general solution more closely while the cantilever beam model starts to deviate noticeably.

2.3. String-Plectrum Interaction

Given a harpsichord string with length L_s , a plucking ratio β , and string tension T , if the amplitude of vibration is small compared to the length of the string, then at the point of contact between the string and the plectrum, the forces on the segment of string with mass Δm can be found:

$$(\Delta m) \frac{\partial^2 x_s(t)}{\partial t^2} = -\frac{T x_s(t)}{L_s \beta (1 - \beta)} + F_{\text{plectrum}_x} \quad (11)$$

$$(\Delta m) \frac{\partial^2 y_s(t)}{\partial t^2} = -\frac{T y_s(t)}{L_s \beta (1 - \beta)} + F_{\text{plectrum}_y} \quad (12)$$

where $x_s(t)$ and $y_s(t)$ denote the string position and F_{plectrum_x} and F_{plectrum_y} are the x and y components of the plectrum force F . The string is not necessarily at the free end of the plectrum, but a distance $L' (< L)$ from the clamped end. Otherwise, this would mean that the string would slip off the plectrum immediately. Here, equations (6) and (7) still apply, so F_{plectrum_x} and F_{plectrum_y} are respectively

$$F_{\text{plectrum}_x} = F \sin \left(\frac{1}{2} \frac{FL'^2}{EI} \right) \quad (13)$$

$$F_{\text{plectrum}_y} = F \cos \left(\frac{1}{2} \frac{FL'^2}{EI} \right) \quad (14)$$

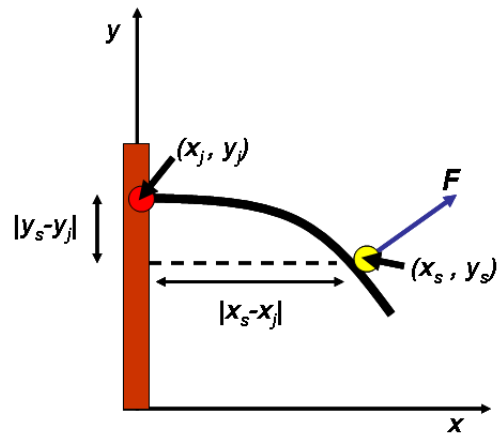


Figure 4: Relative positions of the string and jack.

Table 1: Delrin harpsichord plectrum and steel string values.

Plectrum Parameters	
Length L	6 mm
Width W	2 mm
Thickness H	0.5 mm
Second moment of inertia I	0.029 mm ⁴
Young's modulus E	3 GPa
String Parameters	
Tension T	92 N
Density ρ	7850 kg/m ³
Diameter d	0.37 mm
Linear density μ	0.84 g/m
Length L_s	0.5 m

The clamped end of the plectrum moves with the harpsichord jack, which has position $(x_j, y_j(t))$, as it is constrained to move only in the y direction. According to equation (10), at $l = L'$ the plectrum shape must have

$$x_s(t) - x_j = L' - \left(\frac{F}{EI} \right)^2 \frac{L'^5}{15} \quad (15)$$

$$y_s(t) - y_j(t) = -\left(\frac{F}{EI} \right) \frac{L'^3}{3} + \left(\frac{F}{EI} \right)^3 \frac{L'^7}{105} \quad (16)$$

as shown in Figure 4. Equations (11) to (16) fully describe the motion of the string as it slides along the plectrum. The string slips pass the plectrum when $L' > L$.

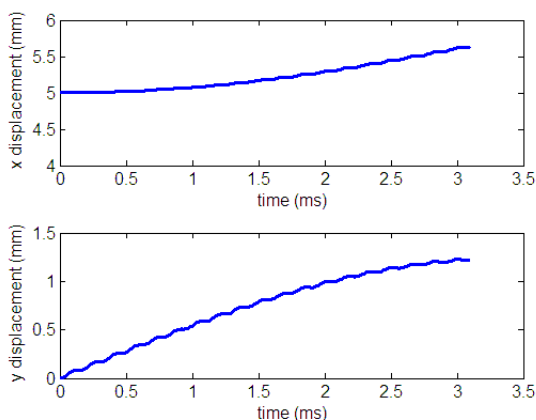
3. RESULTS

3.1. Parameters

Our Delrin harpsichord plectrum and steel string parameters are listed in Table 1. It is common for the harpsichord plectrum's width and thickness to be tapered towards the free end, so the average values were taken.

Table 2: Comparison of plucking ratio and string release amplitude.

$v = 1.0$ m/s	$L' = 5.0$ mm
Plucking Ratio β	Amplitude (mm)
1/2	2.2236
1/3	1.8708
1/4	1.6858
1/5	1.5285
1/6	1.3719
1/7	1.2636

Figure 5: Motion of the string in the horizontal and vertical transverse directions, plucking ratio $\beta = 1/6$ and jack velocity $v_j = 1.0$ m/s.

3.2. Constant velocity case and simulation method

As is investigated in [9, 10, 11], this case refers to when the jack moves at a constant velocity v_j , physically reported to be between 0.1 and 10 m/s. The equation of motion of the jack is then simply

$$y_j(t) = v_j t \quad (17)$$

Equations (11) and (12) were discretized into finite-difference form, with time steps on the order no more than $1 \mu\text{s}$. The plectrum was discretized into spatial samples with a number on the order no less than $N = 10^5$. After each iteration, the corresponding plectrum force F , length along plectrum L' , string velocity (\dot{x}_s, \dot{y}_s) , and the string position (x_s, y_s) were updated.

A simulation was done with a speed value of $v = 1.0$ m/s, initial length from end of plectrum $L' = 5.0$ mm, and the parameters shown in Table 1. As seen in Table 2, it is in agreement with the common experience that as the jack plucks closer to the nut, not only does one perceive tone color changes, but one also notices the immediate decrease in the volume of the note. Results of the simulated string motion with plucking ratio $\beta = 1/6$ and jack velocity $v = 1.0$ m/s are shown in Figure 5. The trajectories of the string exhibit ripples that are results of the oscillatory bending of the plectrum, something that has been observed experimentally and suggested in [10].

4. CONCLUSIONS

In this paper, a more general plectrum model for the harpsichord is presented, one which accounts for large end deflections and also allows for sideways string motion. The physical model of the plectrum follows directly from a more rigorous treatment of the theory of elasticity in bent bars. The model allows for easy further implementation for either digital waveguide or finite-difference string simulations, allowing for excitation in both transverse directions. In the future, friction between the string and the plectrum can be incorporated into the model.

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