

## SPATIAL SOUND SYNTHESIS FOR CIRCULAR MEMBRANES

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### ABSTRACT

Physical models of real or virtual instruments are usually only exploited for the generation of wave forms. However, models of two- and three-dimensional vibrating structures contain also information about the sound radiation into the free field. This contribution presents a model for a membrane from which the required driving functions for a multichannel loudspeaker array are derived. The resulting sound field reproduces not only the musical timbre of the sounding body but also its spatial radiation characteristics. It is suitable for real-time synthesis without pre-recorded or pre-synthesized source tracks.

### 1. INTRODUCTION

The spatial radiation characteristics of musical instruments have been a research topic for quite a long time. Many results have been obtained by careful physical analysis of the acoustical and mechanical properties of musical instruments. Initially, the main intention of this research has been to obtain a scientific understanding of the mysteries behind the sound production of traditional instruments. For a compilation of selected research see e.g. [1].

Recently the focus has shifted from pure understanding to the attempt of actual reproduction of the temporal and spatial properties of real instruments. This change of interest has been driven by new spatial reproduction technologies, like Ambisonics, wave field synthesis, and vector based amplitude panning. With different mathematical, physical, and perceptual methods, these new reproduction techniques overcome the limitations of traditional stereo panning.

However, to use these new reproduction techniques to the best of their possibilities, a more precise knowledge of the spatial directivity of musical instruments is required. During the last years, a lot of effort went into the careful measurement of the spatial radiation characteristics of many traditional musical instruments. Ingenious recording devices have been developed to determine the amount of acoustical energy radiated by musical instruments to the environment in dependence of azimuth and elevation angle and of the musical notes played, see e.g. [2, 3, 4, 5, 6]. Similar to head related transfer functions, this information can be stored and retrieved to reproduce not only the recorded wave form but also the position of a specific instrument in the orchestra setup and the varying pose of the musician during performance. Such an approach might be called data-based, because it relies on recordings of the musical piece as well as of the spatial radiation characteristics of each single instrument. A related problem has been

studied in [7] for the spatial reproduction of non-omnidirectional loudspeakers. Similar to musical instruments, the directivity of the loudspeakers has to be determined from measurements.

A different route is taken here. Rather than relying on measured data, a physical model of a musical instrument is used to derive its spatial characteristics in an analytic way. Such a model can never reproduce all details of a real instrument but it may provide a parametric way for defining and manipulating certain typical directivities.

The physical model used here is not entirely new. It has been used before to reproduce the waveforms of generic vibrating objects (strings, membranes, plates, air columns). In addition to exploiting the physical model for its temporal characteristics only, also its spatial characteristic, i.e. its directivity, is used here.

As an example, a physical model of a membrane is considered. A detailed analysis of the membrane's vibration as well as of the sound propagation from the membrane to the locations of the reproduction loudspeakers allows to determine their driving functions in an analytic fashion. In contrast to a simpler piston model presented earlier [8], the motion of the membrane does not have to be calculated explicitly.

Sec. 2 discusses a simplified membrane model which is sufficient to establish the proposed method. The calculation of the loudspeaker driving functions for wave field synthesis reproduction is briefly shown in Sec. 3. The core results i.e. the link between the physical model and the loudspeaker driving functions are presented in Sec. 4.

### 2. PHYSICAL MODELING OF TWO-DIMENSIONAL STRUCTURES

This section gives a concise introduction to physical modeling with functional transformations. The material discussed here is neither new nor complete. More detailed presentations can be found in [9, 10, 11, 12, 13]. The following analysis is included for completeness and it is restricted to those aspects which are important in the context of spatial sound synthesis. Therefore only a simplified vibration model and only the continuous-time case are considered. These restrictions allow to focus on the calculation of the spatial eigenfunctions, which provide the link to the spatial sound radiation in Sec. 4. A few extensions to more general cases are shortly discussed in Sec. 2.6.

#### 2.1. The Wave Equation

As a simple example of a two-dimensional structure consider the transversal vibrations of a membrane. Its spatial region is described by the vector of two-dimensional spatial coordinates  $\mathbf{x}$ . The type of coordinates (e.g. polar, Cartesian, etc.) matches the

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shape of the spatial region  $\mathbf{x} \in V$  (e.g. a circle, a rectangle, etc.). The vibration of the membrane is described by the deflection  $u(\mathbf{x}, t)$  in the third dimension normal to  $V$ , where  $t$  is the continuous time coordinate. The membrane is fixed to the boundary  $\partial V$  of  $V$ , such that the deflection is zero at the boundary. An excitation  $f(\mathbf{x}, t)$  shall act on the membrane, typically a drum stick, a mallet, or alike. In the simplest case the vibration is governed by the wave equation, which links the excitation  $f(\mathbf{x}, t)$  to the deflection  $u(\mathbf{x}, t)$ , subject to the boundary conditions as

$$\Delta u(\mathbf{x}, t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} u(\mathbf{x}, t) = f(\mathbf{x}, t) \quad \mathbf{x} \in V, \quad (1)$$

$$u(\mathbf{x}, t) = 0 \quad \mathbf{x} \in \partial V, \quad (2)$$

where  $c$  is the speed of sound. The Laplace operator  $\Delta = \nabla^2$  (divergence of the gradient) describes the spatial differentiation in a coordinate free representation. When a specific coordinate system is adopted,  $\Delta$  can be specified with respect to these coordinates.

## 2.2. Fourier-Transformation

Application of the Fourier-transformation with respect to time

$$U(\mathbf{x}, j\omega) = \mathcal{F}\{u(\mathbf{x}, t)\} = \int_{-\infty}^{\infty} u(\mathbf{x}, t) \exp(-j\omega t) dt \quad (3)$$

turns the wave equation into the Helmholtz equation

$$\Delta U(\mathbf{x}, j\omega) + \left(\frac{\omega}{c}\right)^2 U(\mathbf{x}, j\omega) = F(\mathbf{x}, j\omega) \quad \mathbf{x} \in V, \quad (4)$$

$$U(\mathbf{x}, j\omega) = 0 \quad \mathbf{x} \in \partial V. \quad (5)$$

The differentiation theorem of the Fourier-transformation converts the second-order time derivative into a multiplication with  $(j\omega)^2$ . Thus a boundary value problem for the space variable remains.

## 2.3. Sturm-Liouville-Transformation

To remove also the spatial differentiation  $\Delta$ , another transformation for the space variables is required, the so-called Sturm-Liouville-transformation.

### 2.3.1. Definition

Unlike Fourier- or Laplace-transformation, the transformation kernel of the Sturm-Liouville-transformation depends on the spatial differentiation operator (here  $\Delta$ ), the shape of the spatial region (here  $V$ ), and the boundary conditions (5). Therefore, a generic definition with an yet unspecified transformation kernel is given for the moment. The transformation kernel  $K(\beta, \mathbf{x})$  depends on the space variable  $\mathbf{x}$  and a scalar spatial frequency variable  $\beta$ . Then the Sturm-Liouville-transformation  $\mathcal{T}$  is defined by spatial integration over the region  $V$

$$\bar{U}(\beta, j\omega) = \mathcal{T}\{U(\mathbf{x}, j\omega)\} = \int_V U(\mathbf{x}, j\omega) K(\beta, \mathbf{x}) d\mathbf{x}. \quad (6)$$

### 2.3.2. Application to the Helmholtz equation

Application of the Sturm-Liouville-transformation to the Helmholtz equation (4) gives

$$\mathcal{T}\{\Delta U(\mathbf{x}, j\omega)\} + \left(\frac{\omega}{c}\right)^2 \bar{U}(\beta, j\omega) = \bar{F}(\beta, j\omega) \quad (7)$$

Similar to the Fourier-transformation, a differentiation theorem for the transformation of  $\Delta U(\mathbf{x}, j\omega)$  is required.

### 2.3.3. Differentiation Theorem

The procedure for obtaining a differentiation theorem for the Sturm-Liouville-transformation differs from other transformations. E.g. for the Laplace- or the Fourier-transformation, the transformation kernel is known in advance ( $\exp(-st)$  or  $\exp(-j\omega t)$ , respectively). From this transformation kernel, the differentiation theorem is derived, e.g. through integration by parts.

On the other hand, the differentiation theorem for the Sturm-Liouville-transformation is constructed such that it is useful for the problem at hand, here the boundary value problem (4,5). The transformation kernel  $K(\beta, \mathbf{x})$  follows from the procedure for obtaining the differentiation theorem.

This approach is a generalization of the integration by parts. To start with, consider the expression  $U \cdot \nabla K - \nabla U \cdot K$ . Its divergence is given by (the arguments are omitted for simplicity)

$$\nabla(U \cdot \nabla K - \nabla U \cdot K) = U \cdot \Delta K - \Delta U \cdot K. \quad (8)$$

Now integrate both sides over the volume  $V$ . For the divergence on the left hand side, the Gauß-integral-theorem can be applied

$$\int_{\partial V} (U \cdot \nabla K - \nabla U \cdot K) d\mathbf{x} = \int_V U \cdot \Delta K d\mathbf{x} - \int_V \Delta U \cdot K d\mathbf{x}. \quad (9)$$

So far, no assumptions about the transformation kernel  $K$  have been made. To adapt it to the given problem, two conditions are required. One condition concerns only the boundary  $\partial V$ , the other one is valid inside of the spatial region  $V$ .

The first condition requires that  $K(\beta, \mathbf{x})$  shall satisfy the same boundary conditions as  $U(\mathbf{x}, j\omega)$  in (5), i.e.

$$K(\beta, \mathbf{x}) = 0 \quad \mathbf{x} \in \partial V. \quad (10)$$

Due to (5) and (10) the left hand side of (9) vanishes and

$$\int_V U \cdot \Delta K d\mathbf{x} = \int_V \Delta U \cdot K d\mathbf{x} \quad (11)$$

remains. It is the equivalent to integration by parts for homogeneous boundary conditions.

While the first condition for  $K(\beta, \mathbf{x})$  is valid only on the boundary  $\partial V$ , the second one addresses the behaviour inside of  $V$ . It requires that

$$\Delta K(\beta, \mathbf{x}) = (j\beta)^2 K(\beta, \mathbf{x}), \quad \mathbf{x} \in V, \quad (12)$$

such that (11) turns into

$$\int_V \Delta U \cdot K d\mathbf{x} = (j\beta)^2 \int_V U \cdot K d\mathbf{x}. \quad (13)$$

With the definition of the Sturm-Liouville-transformation in (6) follows the differentiation theorem

$$\mathcal{T}\{\Delta U(\mathbf{x}, j\omega)\} = (j\beta)^2 \mathcal{T}\{U(\mathbf{x}, j\omega)\}. \quad (14)$$

Now also the transformation of the first term in (7) can be performed. It removes the spatial differentiation by  $\Delta$  and replaces it by a multiplication with  $(j\beta)^2$ . The result is an algebraic equation

$$(j\beta)^2 \bar{U}(\beta, j\omega) + \left(\frac{\omega}{c}\right)^2 \bar{U}(\beta, j\omega) = \bar{F}(\beta, j\omega). \quad (15)$$

The differentiation theorem (14) can also be established for the more general cases discussed in Sec. 2.6, see e.g. [10, 14, 15].

## 2.4. Transfer Function Description

It is straightforward to solve (15) for the transformation  $\bar{U}(\beta, j\omega)$  of the unknown deflection by simple algebraic operations. The result can be written in the form of

$$\bar{U}(\beta, j\omega) = \bar{G}_u(\beta, j\omega) \bar{F}(\beta, j\omega) \quad (16)$$

with the transfer function

$$\bar{G}_u(\beta, j\omega) = \frac{1}{\left(\frac{\omega}{c}\right)^2 - \beta^2} = \frac{c^2}{\omega^2 - (c\beta)^2}. \quad (17)$$

The roots of this transfer function resemble the dispersion relation of the wave equation where  $\beta$  takes the role of the wave number

$$\beta = \pm \left(\frac{\omega}{c}\right), \quad (18)$$

The transfer function description (16) is the starting point for the derivation of a discrete-time synthesis algorithm which produces time samples of the deflection  $u(\mathbf{x}, t)$  within the accuracy limits imposed by the audio sampling rate. This process involves a continuous-to-discrete-time transformation of  $\bar{G}_u(\beta, j\omega)$  like the impulse-invariant or the bilinear transformation. The impulse-invariant transformation preserves the eigenresonances since it is free from frequency warping. Aliasing can be avoided, if the synthesis is restricted to the audio range which is only reasonable.

The discrete-time synthesis is not developed here because it has been presented in e.g. [10, 16]. The spatial characteristics, which are of interest here, can also be shown in the continuous-time representation. To this end, the spatial structure imposed by the above two conditions for the spatial transformation kernel  $K$  has to be investigated.

## 2.5. Eigenfunctions

### 2.5.1. General Case

The two conditions for the spatial transformation kernel  $K$  from (10) and (12) are compiled here as

$$\Delta K(\beta, \mathbf{x}) + \beta^2 K(\beta, \mathbf{x}) = 0 \quad \mathbf{x} \in V, \quad (19)$$

$$K(\beta, \mathbf{x}) = 0 \quad \mathbf{x} \in \partial V. \quad (20)$$

These conditions constitute a homogeneous boundary value problem for  $K(\beta, \mathbf{x})$  with a similar structure as (4, 5). Boundary value problems of this kind are known as Sturm-Liouville problems and are well studied, see e.g. [14, 15, 17, 18, 19, 20]. Equation (19) can

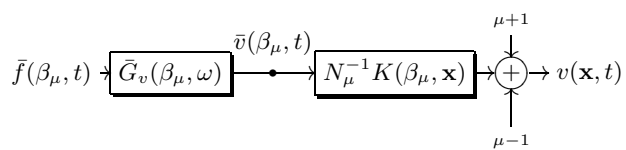


Figure 1: Computational structure in the space-time domain.

be seen as an eigenvalue problem with the eigenvalue  $(j\beta)^2$ . Solutions exist for a discrete set of values for  $\beta = \beta_\mu$  with  $\mu \in \mathbb{Z}$ . The corresponding eigenfunctions form a set of orthogonal functions with

$$\int_V K(\beta_\mu, \mathbf{x}) K(\beta_\nu, \mathbf{x}) = \begin{cases} N_\mu & \mu = \nu \\ 0 & \mu \neq \nu \end{cases}. \quad (21)$$

Thus the inverse Sturm-Liouville-transformation  $\mathcal{T}^{-1}$  has the form of a generalized Fourier series (see (6))

$$U(\mathbf{x}, j\omega) = \sum_\mu \frac{1}{N_\mu} \bar{U}(\beta_\mu, j\omega) K(\beta_\mu, \mathbf{x}). \quad (22)$$

### 2.5.2. Synthesis Structure

By inverse Fourier-transformation follows the deflection of the membrane in the space-time domain

$$u(\mathbf{x}, t) = \sum_\mu \frac{1}{N_\mu} \bar{u}(\beta_\mu, t) K(\beta_\mu, \mathbf{x}). \quad (23)$$

In the same way also the velocity of the membrane can be obtained as the time derivative of the deflection

$$v(\mathbf{x}, t) = \sum_\mu \frac{1}{N_\mu} \bar{v}(\beta_\mu, t) K(\beta_\mu, \mathbf{x}), \quad (24)$$

where  $\bar{v}(\beta_\mu, t)$  is computed similar to (16) with  $\bar{G}_v(\beta_\mu, j\omega)$  instead of  $\bar{G}_u(\beta_\mu, j\omega)$

$$\bar{G}_v(\beta_\mu, j\omega) = j\omega \bar{G}_u(\beta_\mu, j\omega). \quad (25)$$

Fig. 1 shows the structure in the space-time domain, as it evolves from (16), (24), and (25). It consists of a number of parallel branches from (24), where only the one with the number  $\mu$  is shown here. Note that  $\bar{G}_v(\beta_\mu, j\omega)$  is the frequency-domain description of a continuous-time dynamical system. In a discrete-time realization it would be approximated by an IIR filter.

### 2.5.3. Circular Membrane

As a typical example consider a circular membrane with radius  $R$ . It is conveniently described in polar coordinates  $(\rho, \varphi)$  with  $0 \leq \rho \leq R$  and  $0 \leq \varphi < 2\pi$ . The eigenfunctions  $K(\beta, \mathbf{x})$  are then written as  $K(\beta, \mathbf{x}) = K(\beta, \rho, \varphi)$  and the Laplace operator has the form

$$\Delta K(\beta, \rho, \varphi) = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} K(\beta, \rho, \varphi) \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} K(\beta, \rho, \varphi) \quad (26)$$

such that the eigenvalue problem (19) can be written as

$$\rho \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} K(\beta, \rho, \varphi) \right) + \frac{\partial^2}{\partial \varphi^2} K(\beta, \rho, \varphi) + (\rho\beta)^2 K(\beta, \rho, \varphi) = 0. \quad (27)$$

A real-valued form of the solution is

$$K(\beta, \rho, \varphi) = J_n(\rho\beta) \cos n\varphi \quad (28)$$

with the Bessel function  $J_n$  of first kind and order  $n$ . The well-known properties of the Bessel function  $J_n$  show that (28) is a solution of (19) in the form of (27). However, the boundary condition (19) still has to be considered. The boundary  $\partial V$  of a circle is described in polar coordinates by  $\rho = R$ . The boundary condition (19) for the circular membrane thus requires  $K(\beta, R, \varphi) = 0$  which is fulfilled for  $R\beta_{n\nu} = \lambda_{n\nu}$  where  $\lambda_{n\nu}$  is the zero number  $\nu$  of the Bessel function  $J_n$ , i.e.  $J_n(\lambda_{n\nu}) = 0$ . Thus  $\beta$  is restricted to

$$\beta_{n\nu} = \frac{1}{R} \lambda_{n\nu}, \quad n \in \mathbb{Z}, \quad \nu = 1, 2, \dots \quad (29)$$

and the eigenfunctions can be indexed in the order of the Bessel function  $n$  and the number of its zeros  $\nu$  as

$$K(\beta_{n\nu}, \rho, \varphi) = J_n \left( \frac{\rho}{R} \lambda_{n\nu} \right) \cos n\varphi = J_n(\rho\beta_{n\nu}) \cos n\varphi. \quad (30)$$

The double index  $n$  and  $\nu$  is not convenient, but it cannot be avoided altogether. As a link to the audio frequencies consider the dispersion relation (18) with

$$\omega_{n\nu} = 2\pi f_{n\nu} = c\beta_{n\nu} = \frac{c}{R} \lambda_{n\nu}. \quad (31)$$

The lowest eigenfrequency corresponds to the first zero of  $J_0$  at  $\lambda_{0,1} = 2.404826$  with  $n = 0$  and  $\nu = 1$  as

$$f_{0,1} = \frac{1}{2\pi} \frac{c}{R} \lambda_{0,1}. \quad (32)$$

For  $c = 340 \frac{\text{m}}{\text{s}}$  and  $R = 0.54 \text{ m}$  follows  $f_{0,1} \approx 240 \text{ Hz}$ .

A simpler indexing scheme is obtained by ordering the values  $\beta_\mu = \beta_{n\nu}$  in increasing order of the resulting audio frequencies (32). Both indexing schemes are used in the sequel.

## 2.6. Extensions

The concise presentation of the physical modeling method in this section has been chosen because it provides a short and complete route from the initial problem to the structure of the eigenfunctions (28) and their time evolution expressed by the coefficients  $\bar{u}(\beta_\mu, t)$  and  $\bar{v}(\beta_\mu, t)$  in (23) and (24), respectively. However the functional transformation approach is applicable to far more general problems. A few extensions are highlighted below.

### 2.6.1. Damping and Dispersion

More accurate models for vibrating bodies include also effects like damping or dispersion. Their physical description results in additional terms in the partial differential equation (1). The general form of the eigenfunction remains but the functions  $\bar{v}(\beta_\mu, t)$  decay through damping and the eigenvalues  $\beta_\mu$  are shifted through damping and dispersion.

### 2.6.2. Non Self-Adjoint Problems

The spatial differential operator  $\Delta$  of the wave equation (1) appears on both sides of (11). Boundary value problems with this symmetry are called self-adjoint. They are a special case of more general spatial differential operators  $L$  comprising more complex spatial differentiation terms. In the general case, a relation similar to (11) holds, with an operator  $L$  on one side and its adjoint operator  $\tilde{L}$  on the other. Two different sets of eigenfunctions  $K$  and  $\tilde{K}$  result which are bi-orthogonal to each other.

### 2.6.3. Other Types of Boundary Conditions

The so-called Dirichlet boundary conditions in (2) and (20) are not the only case where the left-hand side of (9) vanishes. This would also be the case for Neumann boundary conditions, i.e.  $\nabla U = \mathbf{0}$  and  $\nabla K = \mathbf{0}$  on the boundary  $\mathbf{x} \in \partial V$ . Also impedance type boundary conditions (Robin boundary conditions) are possible. For non self-adjoint problems, different boundary conditions for  $K$  and  $\tilde{K}$  are required.

### 2.6.4. Other Spatial Shapes

The formulation in Sec. 2.5.1 and Sec. 2.5.2 does not imply a certain spatial shape, a specific coordinate system, or a fixed number of spatial dimensions. Therefore the representations (23) and (24) are valid for all shapes of practical interest. Practical difficulties may arise in the analytical calculation of the eigenfunctions and eigenvalues. For certain standard shapes (e.g. rectangular, circular, elliptical, spherical, and others) they can be found by separation of variables.

## 3. SPATIAL SOUND REPRODUCTION

### 3.1. Overview

Two-channel stereophony and common surround sound formats for sound reproduction exploit the spatial hearing capabilities of human perception. Phantom sources are placed by applying panning or other spatial effects. Besides these common formats, several massive multichannel reproduction methods have gained acceptance. So far they are no alternative for home users, but the number of installations in cinemas, laboratories, and other professional acoustic spaces is increasing. Common spatial sound reproduction methods are Ambisonics, wave field synthesis, and vector based amplitude panning, see e.g. [21] for a description and references to the original literature. The focus in this contribution is on wave field synthesis, although an extension to Ambisonics would appear to be feasible as well.

### 3.2. Wave Field Synthesis

Consider a planar loudspeaker array for reproduction with wave field synthesis. No special shape is assumed, but rectangles or circles are typical shapes. The position of an arbitrary loudspeaker is  $\mathbf{x}_0$  and the vector normal to the array at  $\mathbf{x}_0$  is called  $\mathbf{n}_0$ . The normal vector points inwards, e.g. for a circular array it points to the center of the circle. To calculate the driving functions for the loudspeakers, the sound field at their positions must be known. It can be obtained either by suitable processing of spatial recordings of an acoustic event or from a model of an acoustic scene. The

model-based view is adopted here, where the membrane model from Sec. 2 serves as a description of a spatially extended source.

The calculation of the loudspeaker driving functions from a piston model has already been discussed in [8]. Therefore only a short account is given here for the sake of completeness.

The Fourier spectrum  $D(\omega, \mathbf{x}_0)$  of the driving function for a loudspeaker at the position  $\mathbf{x}_0$  is derived in [22, ch. 13.2][23] as

$$D(\omega, \mathbf{x}_0) = w(\mathbf{x}_0, \mathbf{x}_S)A(\mathbf{x}_0)H_{wfs}(\omega) \mathbf{n}_0^T \nabla P(\omega, \mathbf{x}_0). \quad (33)$$

where

- $w(\mathbf{x}_0, \mathbf{x}_S)$  is a spatial window which selects the active loudspeakers for a certain source position  $\mathbf{x}_S$ ,
- $A(\mathbf{x}_0)$  is an amplitude factor,
- $H_{wfs}(\omega)$  is a frequency selective filter which is independent of the loudspeaker position.

The essential component to determine the loudspeaker driving functions is the gradient of the sound pressure  $\nabla P(\omega, \mathbf{x}_0)$ . Its derivation from the membrane model in Sec. 2 is shown in the following section.

#### 4. SPATIAL SOUND SYNTHESIS

This section derives the loudspeaker driving functions for a wave field synthesis array according to Sec. 3 from the sound synthesis model of a membrane according to Sec. 2. Techniques similar to the piston model from [8] are applied, but instead of a rigid piston the eigenfunctions of a vibrating membrane from (24) are considered. The result is a matrix description where each entry indicates the contribution of a specific eigenfunction to a specific loudspeaker of the reproduction array.

##### 4.1. Membrane Coordinate System

So far no special type of spatial coordinates has been assumed. Also no distinction has been made between the coordinates of the membrane model and the coordinates for the loudspeaker array. For practical reasons, it is of advantage to use two separate coordinate systems for the membrane and for the array. The position of the membrane with respect to the array is not fixed and the membrane might change its position or its orientation with respect to the array. Therefore a coordinate system suitable for a circular membrane is introduced first, the connection to a loudspeaker array is then shortly discussed in Sec. 4.7. The spatial coordinate system for the membrane is shown in Fig. 2.

The coordinates for the membrane are denoted with greek letters. The components of the vector of space coordinates  $\xi$  are

$$\xi = [\xi \quad \eta \quad \zeta]^T. \quad (34)$$

The circular membrane resides in the center of the  $\xi$ - $\eta$ -plane with

$$\xi = \rho \cos \varphi \quad (35)$$

$$\eta = \rho \sin \varphi \quad (36)$$

$$\zeta = 0 \quad (37)$$

The loudspeaker position for the calculation of the sound pressure gradient is an arbitrary location

$$\xi_0 = [\xi_0 \quad \eta_0 \quad \zeta_0]^T \quad (38)$$

with the assumption that the distance  $\zeta_0$  from the  $\xi$ - $\eta$ -plane is large compared to the membrane radius, i.e.  $\zeta_0 \gg R$ .

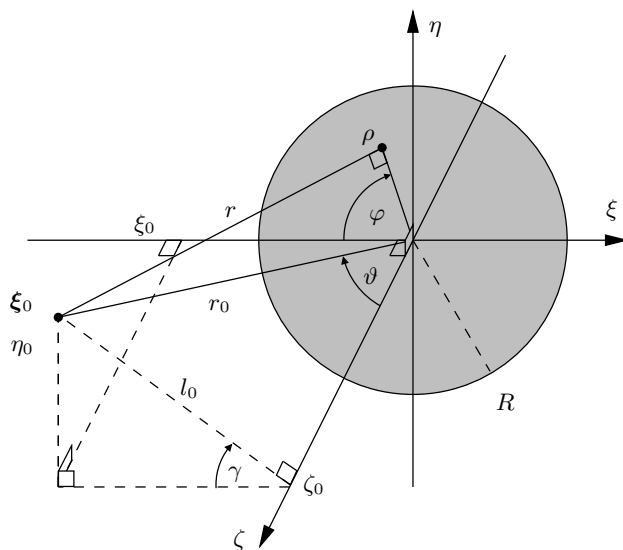


Figure 2: 3D spatial coordinate system for the membrane model. Small rectangles indicate right angles.

##### 4.2. Determination of the Sound Pressure Gradient

The sound pressure gradient  $\nabla P$  according to (33) is now determined in the temporal frequency domain. At first, Fourier transformation with respect to time for the velocity from (24) gives

$$V(\xi, \omega) = \sum_{\mu} \frac{1}{N_{\mu}} \bar{V}(\beta_{\mu}, \omega) K(\beta_{\mu}, \xi) \quad \xi \in V. \quad (39)$$

The sound pressure at the location  $\xi_0$  is obtained similar to the piston model discussed in [8] as

$$P(\xi_0, \omega) = j\omega\rho_L \int_V V(\xi, \omega) G(\xi|\xi_0, \omega) d\xi, \quad (40)$$

where  $G(\xi|\xi_0, \omega)$  is the Green's function in the temporal frequency domain for the propagation from a point on the membrane  $\xi$  to the arbitrary location  $\xi_0$ . Note that in contrast to the piston model, the velocity is different for all points  $\xi \in V$  and thus has to be part of the integration. However, it is not very attractive to perform the spatial integration in (40) over the vibrating membrane for every time instant.

Instead, the sound pressure  $P(\xi_0, \omega)$  can be expressed by the spectral coefficients  $\bar{V}(\beta_{\mu}, \omega)$  from (39) by inserting into (40) as

$$P(\xi_0, \omega) = \sum_{\mu} \frac{1}{N_{\mu}} \bar{V}(\beta_{\mu}, \omega) H(\beta_{\mu}, \xi_0) \quad (41)$$

with

$$H(\beta_{\mu}, \xi_0) = j\omega\rho_L \int_V K(\beta_{\mu}, \xi) G(\xi|\xi_0, \omega) d\xi. \quad (42)$$

Each function  $H(\beta_{\mu}, \xi_0)$  describes the contribution of the eigenfunction with number  $\mu$  to the sound pressure induced by the membrane at the location  $\xi_0$ . Note that  $H(\beta_{\mu}, \xi_0)$  is independent of

the actual state of the membrane. Similar to the eigenfunctions  $K(\beta_\mu, \xi)$ , also  $H(\beta_\mu, \xi_0)$  can be calculated in advance.

For the sound pressure gradient at the location  $\xi_0$  according to (33) the gradient has to be calculated with respect to  $\xi_0$ . Then (41) turns into

$$\nabla P(\xi_0, \omega) = \sum_{\mu} \frac{1}{N_{\mu}} \bar{V}(\beta_{\mu}, \omega) \nabla H(\beta_{\mu}, \xi_0) \quad (43)$$

with

$$\nabla H(\beta_{\mu}, \xi_0) = j\omega\rho_L \int_V K(\beta_{\mu}, \xi) \nabla G(\xi|\xi_0, \omega) d\xi. \quad (44)$$

Fig. 3 shows the resulting computational structure for the sound pressure gradient. The structure on the top is an extension of the membrane model from Fig. 1 by an integration similar to (40). Its disadvantage is that each point  $\xi$  (or a sufficiently dense grid) has to be calculated in order to perform the spatial integration with  $\nabla G(\xi|\xi_0, \omega)$ . The structure on the bottom results from the simplification by obtaining the sound pressure gradient directly from  $\bar{V}(\beta_{\mu}, \omega)$  with (43). The functions  $\nabla H(\beta_{\mu}, \xi_0)$  are called the *transfer coefficients* from the eigenfunction  $\mu$  to the location  $\xi_0$ .

The sound pressure gradient  $\nabla P(\xi_0, \omega)$  can be calculated from (43) once the transfer coefficients according to (44) are known. Since the eigenfunctions  $K(\beta_{\mu}, \xi)$  for a circular membrane are already given by (30) it remains to determine the gradient of the Green's function  $\nabla G(\xi|\xi_0, \omega)$ .

### 4.3. Gradient of the Green's Function

The integration in (44) collects the contributions of all point-like regions inside of the membrane area  $V$ . Therefore the Green's function of a point source is chosen as propagation model

$$G(\xi|\xi_0, \omega) = \frac{1}{2\pi r} e^{-jkr}, \quad r = |\xi_0 - \xi|, \quad k = \frac{\omega}{c}. \quad (45)$$

The distance  $r$  from an arbitrary point on the membrane (formally  $\xi \in V$ ) to the arbitrary loudspeaker position  $\xi_0$  is given in detail by (see (35–37) and Fig. 2)

$$r(\xi_0) = \sqrt{(\xi_0 - \xi)^2 + (\eta_0 - \eta)^2 + \zeta_0^2}. \quad (46)$$

The distance  $r(\xi_0)$  is written as a function of  $\xi_0$  because the gradient is calculated with respect to the loudspeaker location  $\xi_0$ .

Application of the chain rule of derivation to (45) gives

$$\nabla G(\xi|\xi_0, \omega) = \frac{\partial}{\partial r} G(\xi|\xi_0, \omega) \nabla r. \quad (47)$$

From (45) and (46) follows with simple calculation rules

$$\frac{\partial}{\partial r} G(\xi|\xi_0, \omega) = -(1 + kr) \frac{1}{r} \nabla G(\xi|\xi_0, \omega) \quad (48)$$

$$\nabla r = \frac{1}{r} (\xi_0 - \xi) \quad (49)$$

such that

$$\begin{aligned} \nabla G(\xi|\xi_0, \omega) &= -(1 + kr) \frac{1}{r^2} G(\xi|\xi_0, \omega) (\xi_0 - \xi) \\ &= -\frac{1 + jkr}{2\pi r^3} e^{-jkr} (\xi_0 - \xi). \end{aligned} \quad (50)$$

Due to (46)  $\nabla G(\xi|\xi_0, \omega)$  turns out to be a complicated function of  $\xi_0$  and  $\xi$ . Therefore some approximations are required before the integration in (44) can be carried out efficiently.

### 4.4. Approximations for the Green's Function

Some simplifications are permitted since the loudspeaker position has been assumed to be somewhat remote from the membrane (see Sec. 4.1). However, care has to be taken not to oversimplify the problem. The derivation below is known from a simplified piston model [24, 25]; here it is applied to the membrane model discussed in Sec. 2.

For the magnitude terms in (50) it is suitable to replace the distance  $r = |\xi_0 - \xi|$  by the distance to the center of the membrane  $r_0 = |\xi_0|$  i.e.

$$r_0(\xi_0) = |\xi_0| = \sqrt{\xi_0^2 + \eta_0^2 + \zeta_0^2}. \quad (51)$$

Applying the same approximation to the phase term in (50) would reduce the whole membrane model to a point source (see [8]). For a more detailed approximation the distance  $r$  is formulated with (46) as

$$r^2 = (\xi_0^2 + \eta_0^2 + \zeta_0^2) - 2(\xi_0\xi + \eta_0\eta) + \xi^2 + \eta^2 \quad (52)$$

and rewritten with the polar coordinates (35–36)

$$r^2 = r_0^2 - 2\rho(\xi_0 \cos \varphi + \eta_0 \sin \varphi) + \rho^2. \quad (53)$$

The second term on the right hand side is a mixture of Cartesian  $(\xi_0, \eta_0)$  and polar coordinates  $(\rho, \varphi)$ . Expressing  $\xi_0$  and  $\eta_0$  also in polar form with the magnitude  $l_0$  and the angle  $\gamma$  (see Fig. 2)

$$l_0 = \sqrt{\xi_0^2 + \eta_0^2}, \quad \frac{\eta_0}{\xi_0} = \tan \gamma \quad (54)$$

results in

$$\xi_0 \cos \varphi + \eta_0 \sin \varphi = l_0 \cos(\varphi - \gamma) \quad (55)$$

and

$$r^2 = r_0^2 - 2\rho l_0 \cos(\varphi - \gamma) + \rho^2. \quad (56)$$

In this form, it is easy to recognize the dependency on  $\rho$ . Neglecting the second order term and keeping the first order term gives

$$r^2 \approx r_0^2 - 2\rho l_0 \cos(\varphi - \gamma). \quad (57)$$

Further approximations are possible by writing  $r$  as

$$r(\rho, \varphi) = r_0 \sqrt{1 - 2 \frac{\rho}{r_0} \frac{l_0}{r_0} \cos(\varphi - \gamma)}. \quad (58)$$

With  $\rho \ll r_0$ ,  $l_0 \leq r_0$ , and  $\sqrt{1-x} \approx 1 - \frac{1}{2}x$  follows

$$r(\rho, \varphi) \approx r_0 - \rho \frac{l_0}{r_0} \cos(\varphi - \gamma). \quad (59)$$

The fraction  $l_0/r_0$  can be expressed as

$$\frac{l_0}{r_0} = \sqrt{1 - \left(\frac{\zeta_0}{r_0}\right)^2} = \sin \vartheta. \quad (60)$$

The angle  $\vartheta = \vartheta(\xi_0)$  depends only on the position  $\xi_0$  of the loudspeaker and is shown in Fig. 2.

The distance  $r$  in the exponential term of (50) can finally be approximated by

$$r(\rho, \varphi) \approx r_0 - \rho \sin \vartheta \cos(\varphi - \gamma). \quad (61)$$

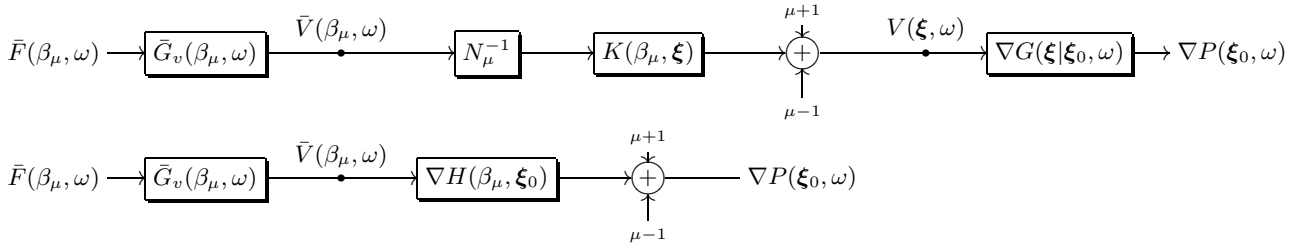


Figure 3: Computational structures for the sound pressure gradient. *Top*: Cascade of the membrane model from Fig. 1 and the integration with the gradient of the Green's function  $\nabla G(\xi|\xi_0, \omega)$ . *Bottom*: Simplification through the transfer coefficients  $\nabla H(\beta_\mu, \xi_0)$  from (44).

Inserting into (50) and collecting terms gives

$$\nabla G(\xi|\xi_0, \omega) \approx \nabla G(\mathbf{0}|\xi_0, \omega) \exp(jk\rho \sin \vartheta \cos(\varphi - \gamma)). \quad (62)$$

This approximation for  $\nabla G(\xi|\xi_0, \omega)$  can be interpreted as the gradient of the Green's function  $\nabla G(\mathbf{0}|\xi_0, \omega)$  from the center of the membrane to  $\xi_0$  and an exponential directivity term which considers the position  $(\rho, \varphi)$  on the circular membrane. The directivity term affects only the phase but not the magnitude due to the different approximations (51) and (61).

#### 4.5. Determination of the Transfer Coefficients

In the approximation for  $\nabla G(\xi|\xi_0, \omega)$  in (62), the dependency on the polar coordinates of the membrane,  $\rho$  and  $\varphi$ , is restricted to the arguments of elementary functions (exponential and trigonometric functions). Therefore the integration in (44) can be carried out partly in closed form. Writing the integration in (44) in polar coordinates gives

$$\nabla H(\beta_\mu, \xi_0) = j\omega\rho_L \int_0^R \int_0^{2\pi} K(\beta_\mu, \xi) \nabla G(\xi|\xi_0, \omega) \rho d\varphi d\rho. \quad (63)$$

The argument  $\xi$  of  $K$  and  $\nabla G$  has to be understood also in polar form according to (35–36).

Inserting the approximation for  $\nabla G(\xi|\xi_0, \omega)$  from (62) gives an approximation  $\nabla \hat{H}$  for the transfer coefficients (63) and the eigenfunctions  $K(\beta_\mu, \xi)$  from (30)

$$\nabla \hat{H}(\beta_\mu, \xi_0) = j\omega\rho_L \nabla G(\mathbf{0}|\xi_0, \omega) \int_0^R J_n(\rho\beta_{n\nu}) Q_1 \rho d\rho. \quad (64)$$

with  $Q_1 = Q_1(\rho, \varphi, \xi_0, n, \nu)$

$$Q_1 = \int_0^{2\pi} \exp(jk\rho \sin \vartheta \cos(\varphi - \gamma)) \cos n\varphi d\varphi. \quad (65)$$

From the definition of the Bessel functions  $J_n$  follows the identity

$$\int_0^{2\pi} e^{jx \cos(\varphi - \gamma)} \cos n\varphi d\varphi = j^n 2\pi J_n(x) \cos n\gamma, \quad (66)$$

such that the integral in (65) can be expressed in closed form

$$Q_1(\rho, \varphi, \xi_0, n, \nu) = j^n 2\pi J_n(k\rho \sin \vartheta) \cos n\gamma. \quad (67)$$

For the approximate transfer coefficients  $\nabla \hat{H}$  follows from (63)

$$\nabla \hat{H}(\beta_\mu, \xi_0) = 2\pi j^{n+1} \omega\rho_L \cos n\gamma \nabla G(\mathbf{0}|\xi_0, \omega) Q_2(\xi_0, n, \nu, R) \quad (68)$$

with

$$Q_2(\xi_0, n, \nu, R) = \int_0^R J_n(\rho\beta_{n\nu}) J_n(k\rho \sin \vartheta) \rho d\rho. \quad (69)$$

This integral over a finite range cannot be expressed by analytical terms and has to be evaluated numerically. However, it depends only on indices  $n$  and  $\nu$  of the respective eigenfunction, the fixed location  $\xi_0$  of the loudspeaker, and the radius  $R$  of the membrane.

The other terms in (68) can be evaluated in closed form. The term  $\nabla G(\mathbf{0}|\xi_0, \omega)$  follows from (50) with  $\xi = 0$  and consequently  $r = r_0$  (see (52)). The term  $\cos n\gamma$  can be expressed by the Chebyshev Polynomials  $T_n(x)$  of order  $n$  and with (54) as

$$\cos n\gamma = T_n(\cos \gamma), \quad \cos \gamma = \left(1 + \left(\frac{\eta_0}{\xi_0}\right)^2\right)^{-\frac{1}{2}}. \quad (70)$$

Thus the transfer coefficients  $\nabla \hat{H}$  can be evaluated by the equations (68) – (70). Now all components in the spatial sound synthesis model in Fig. 3 are determined and an approximation of the sound pressure gradient  $\nabla P$  at an arbitrary location  $\xi_0$  can be calculated from the physical model of the membrane.

#### 4.6. Spatial Sound Synthesis Structure

So far it has been shown how to obtain the sound pressure gradient at an arbitrary location within the limits of reasonable approximations. In principle this process can be carried out for different loudspeaker positions  $\xi_m$  as required by a specific wave field synthesis arrangement. Then the above process yields a matrix of transfer coefficients, where the entry  $(\mu, m)$

$$\nabla \hat{H}_{\mu, m} = \nabla \hat{H}(\beta_\mu, \xi_m) \quad (71)$$

determines the contribution of the eigenfunction  $\mu$  to the loudspeaker position  $m$ . The driving function  $D(\omega, \xi_m)$  for this loudspeaker results from (33).

#### 4.7. Membrane and Array Coordinates

For practical reasons it is convenient to use different coordinate systems for the membrane and for the wave field synthesis array.

The membran coordinates  $\xi$  have been introduced in Sec. 4.1 and in Fig. 2. A system of array coordinates  $\mathbf{x}$  can be chosen to suit the spatial shape of the loudspeaker array. The link between both coordinate systems is provided by a matrix description of a rotation and a translation. Details for a circular array are given in [8]. The choice of two different coordinate systems makes it easier to change the position and the orientation of the membrane with respect to the reproduction array.

## 5. CONCLUSION

This contribution presented a combined physical model for reproducing both the waveform and the spatial radiation characteristics of a sounding membrane. The focus of the development was the calculation of the transfer coefficients from each eigenfunction of the membrane to each loudspeaker of the array. These transfer coefficients are the link between physical modelling sound synthesis on the one hand and wave field synthesis on the other hand. The key feature of this approach is that any numerical integration over the surface of the membrane can be avoided.

This procedure is not restricted to the simple membrane model used as illustration here. Several extensions have been already mentioned in Sec. 2.6. On the other hand, physical modelling sound synthesis is not necessarily connected to wave field synthesis. A model-based determination of the Ambisonics encoding process should be possible along the same lines.

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