MORE MODAL FUN — "FORCED VIBRATION" AT ONE POINT

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ABSTRACT

The question, if a vibrating object can be forced to follow a given movement profile at one point forms a case of an inverse problem. It is shown that for the specific setting of an object described by modal data, this question may be solved by a newly developed method. The new technique has several strengths, such as allowing to compute modal data for the constrained scenario and forming a basis for precise and stable simulations. The latter potential is shown at a short example, a stiff string being hammered against a fixed board by a hammer of infinite mass.

1. INTRODUCTION

Solving or understanding problems in acoustics usually involves handling a differential equation with boundary conditions, representing a physical system, with certain terms of external influences such as forces, pressures or potentials. If such a system of equations fully describes a problem of classical physics there will be one *unique* solution for a given "allowed input (force) term", a fact that is often implicitely assumed and not explicitely shown with mathematical strictness. (Well-defined *initial conditions* are furthermore generally neccessary for the solution to be unique. These general thoughts will be looked at more closely in mathematical terms in following sections.)

Practical problems sometimes lead to a mathematical question of the same general structure as just described but involving an *unknown* "force term" and additional constraints on the solution. As an example it may be very difficult to measure the exact forces acting on a string struck by a hammer or plucked with whatever mechanism, since any direct measurement of the force at the region of contact (e.g. between piano hammer and string or finger and string) inevitably alters the whole process of interest. In such a situation it may be much easier to measure the string *displacement* at one point with good temporal and spatial precision (e.g. by means of a pickup or high speed camera). The question therefore arises if for such a measured displacement profile the causing force signal may be reconstructed. (After the above mentioned uniqueness of solution reconstructing the force signal is equivalent to determining the evolution of the complete state of the system, here the string.) The same kind of mathematical question may occur in computational simulations when trying to simplify the structure of the system equations by considering certain couplings as "one-way": As an example, for simulating the "sympathetic vibration" of neighbouring strings induced through a vibrating soundboard it may be plausible to consider the impedance of the soundboard with respect to resonating strings as infinitely high. In this case one must demand for the resonating strings to "strictly follow" the movement of the soundboard at the point of contact (the bridge). Again the underlying force acting on the strings in this setting is a priori not known. Moreover in this case it is not a priori clear if the resulting system of equations has a solution at all, since starting point is here a hypothetical displacement curve which, in contrast to the previous situation, does *not* originate from measurements at an existing physical system. Moreover, even in the first case measurements are of limited precision so that the mathematical question of existence of a solution indeed is relevant — in addition to (and as a necessary precondition to) deriving a practical receipt for finding one.

Intuitively the described approach of starting from a given displacement profile, not a force signal, may appear as a way of "reducing the degrees of freedom". (In the concrete mathematic formulation presented in the following this will indeed turn out to be the case). One might therefore expect the presented questions (of existence and nature of solutions to a system of equations with a given "displacement profile" and unkown "input force") not to be too difficult to answer. However the roughly described situations represent a class of mathematical problems sometimes denoted as "inverse problems" which only rather recently has received systematic attention and may turn out to be quite challenging. In this contribution we look at a specific mathematical setting whose choice is shortly motivated in the next subsection. The following sections derive how the question described above may be formulated in this concrete mathematical context, prove the existence and uniqueness of the solution and present a practical receipt for solving it. The practicability of the presented method is shortly demonstrated at a simple example.

1.1. Linear evolution equations

Many scenarios of acoustic vibration around an equilibrium state [1] can be written in form of an "evolution equation"

$$\dot{\vec{z}}(t) = A\vec{z}(t) + \vec{f}(t), \tag{1}$$

where $\vec{z}(t) \in Z$ represents the state of the system at time t in its *state (vector) space* Z and $A : Z \to Z$ is an operator on Z. The vector $\vec{f}(t)$ represents external forces (or other influences such as pressures or potentials). Equation (1) is very general; all specific complexity is here hidden in the extact structure of the state space Z and of course the operator A. Often — mostly for "sufficiently small" deviations from equilibrium — A may be chosen linear and compact. Many important equations in acoustics (such as the wave

[1] Apart from situations that typically come to mind here, such as vibrating strings, beams, membranes, plates. . . , also scenarios of wave propagation in finite domains may be looked at in this way, altough they may rarely be expressed exactly in such a form.

equation in a specific 1-, 2- or 3-dimensional domain with certain boundary conditions) are usually not directly written in a form (1), but, while an equivalent formulation of this form is hardly ever presented, this would be possible. (Such reformulations may however involve substantial efforts of mathematical theory.) The present work starts with a specific version of equation (1) since this form somewhat presents the most general basis and justification(!) for mathematical tools such as Eigenvalue analysis which are highly important and commonly used (sometimes implicitely) for handling acoustic equations (see e.g. [1]).

Returning to initial thoughts given above, it is known [2] that for A being linear and compact, equation (1) *has one unique solution* for any *initial state* $\vec{z}_0 := \vec{z}(0)$ at time 0. The most compressed way of expressing this fact is probably by writing down said analytical solution in the form

$$\vec{z}(t) = e^{tA}(\vec{z}_0 + \int_0^t e^{-sA}\vec{f}(s)ds). \tag{2}$$

This direct expression is however not of high pratical use without further analysis of the structure of the operator A, such as its expression in terms of Eigenvalues and -vectors. It may however come handy to be aware of the existence of this expression.

2. FORCED MOVEMENT AT ONE POINT OF AN OBJECT IN MODAL DESCRIPTION

Following motivations just given, the kind of questions posed at the beginning of the introduction ("given displacement profile, unknown input force") shall be formulated mathematically in the context of equation (1). For a system described by such an equation the *observed* displacement $y(t)$ (or whatever variable of observation, e.g. velocity or pressure) shall be characterised by another *linear* dependency on the state $\vec{z}(t)$:

$$y(t) = B\vec{z}(t). \tag{3}$$

Now, given the value of such a known observation variable $y(t)$ over time (weither originating from measurement or from assumptions in a simulation) for a system characterised by operators A and B and equations (1) and (3) with an unknown force term $\vec{f}(t)$ and an initial state $\vec{z}_0 = \vec{z}(0)$, does a solution $\vec{z}(t)$ of (1) and (3) exist and how can it be determined? As already noted, and as seen from (2), asking for a solution $\vec{z}(t)$ is equivalent to the question of a "suitable" underlying force vector $\vec{f}(t)$. It is easily understood that the existence of a solution and the dimension of the "solution space" must depend on the exact shapes of $y(t)$, A, B: as most simple examples, for $B = 0$ (the zero operator) and $y(t) \neq 0$ for any one t no solution can exist, while for $B = 0$ and $y(t) = 0 \forall t$ any $\vec{f}(t)$ and resulting $\vec{z}(t)$ will satisfy the equations. In the following subsections a specific concrete setting of A and B according to the kind of concrete problems depicted in the introduction will be specified and the posed question will be solved for this concrete case.

2.1. Mathematical formulation of the problem

As mentioned, many problems of acoustics can be formulated with a compact operator A; such operators have a discrete, countable spectrum [3] which allows for a convenient characterisation of the unique solution (2) in terms of "modes of vibration". For the following we assume that the involved vibrating object is already represented in such a modal description. This starting point has the advantage of being very general: the method developed in the following may be applied with modal data gained in whatever way — from theoretical analysis of a differential equation, from numeric analysis of a finite-dimensional problem (finite-element data, individual "lumped" element model) or from experimental measurements. We further assume that only a finite number of modes need to be considered, an assumption that is motivated by perceptual aspects (modal frequencies above the hearing range may be neglected) as well as for reasons of handling (matrix diagonalisation) by means a finite computing architecture. (A large part of the following arguments actually apply even for an infite but countable number of discrete modes, which is expressed by leaving blank the range of summations. At a certain point however, eigenvalues and -vectors have to be computed by means of a numeric algorithm which of course requires finite-dimensional data.)

As described in many textbooks (compare e.g. [4]) every mode of vibration is characterised by a frequency of vibration and a damping factor, or, equivalently, factors of stiffness k and friction r and "modal shapes" manifested by weighting factors at each point of interest. Linear operators on a finite-dimensional (state) space (as is the case for a finite number of modes) are commonly represented by matrices, and in the just described modal basis A takes the form

$$A = \begin{pmatrix} A_1 & 0 & 0 & \dots \\ 0 & A_2 & 0 & \dots \\ 0 & 0 & \ddots & \\ \vdots & \vdots & & \ddots \end{pmatrix}, \tag{4}$$

with

$$A_n := \begin{pmatrix} 0 & 1 \\ -k_n & -c_n \end{pmatrix}, \tag{5}$$

The state vector $\vec{z}$ here contains "modal displacements" and "modal velocities": $\vec{z} = (x_1, \dot{x}_1, x_2, \dot{x}_2, \dots)^{\mathrm{t}}$ (written as a column vector).

The displacement of the object at one point y is then a weighted sum of the modal displacements: $y(t) = \sum w_n x_n(t)$. In the same way, for a force $f(t)$ acting at one point of contact the force vector $\vec{f}$ is given as $\vec{f}(t) = f(t) \cdot (0, w_1, 0, w_2, \dots)^{\mathrm{t}}$. By introducing the column vectors

$$\vec{w}_n := (0, w_n)^{\mathrm{t}},\ \vec{w} := (\vec{w}_1^{\mathrm{t}}, \vec{w}_2^{\mathrm{t}}, \dots)^{\mathrm{t}} = (0, w_1, 0, w_2, \dots)^{\mathrm{t}} \tag{6}$$

and

$$\tilde{\vec{w}}_n := (w_n, 0)^{\mathrm{t}},\ \tilde{\vec{w}} := (\tilde{\vec{w}}_1^{\mathrm{t}}, \tilde{\vec{w}}_2^{\mathrm{t}}, \dots)^{\mathrm{t}} = (w_1, 0, w_2, 0, \dots)^{\mathrm{t}} \tag{7}$$

these relations may be written in matrix notation as

$$y(t) = \tilde{\vec{w}}^{\mathrm{t}}\vec{z}(t) \tag{8}$$

and

$$\vec{f}(t) = \vec{w}f(t) \tag{9}$$

The initial question, for an object in modal description subject to an unknown external force at one point at which the displacement over time is given now reads in the introduced notation as:

Question

For a given displacement $y(t)$ over time according to (8) and an operator A as in (4) and (5), can equation (1) be solved with a suitable force profile $f(t)$ acting according to (9)?

2.2. An evolution equation for the forced vibration setting

Several relations which can be easily verified will be useful in the following:

$$\tilde{\vec{w}}^{\mathrm{t}}\vec{w} = 0,\ (\tilde{\vec{w}}_n^{\mathrm{t}}\vec{w}_n = 0 \forall n) \tag{10}$$

$$\tilde{\vec{w}}^{\mathrm{t}}A = \vec{w}^{\mathrm{t}},\ (\tilde{\vec{w}}_n^{\mathrm{t}}A_n = \vec{w}_n^{\mathrm{t}} \forall n) \tag{11}$$

If all stiffness factors are nonzero (which we assume), $k_n \neq 0 \forall n$, all A_n and also A are invertible:

$$A_n^{-1} = \begin{pmatrix} -c_n/k_n & -1/k_n \\ 1 & 0 \end{pmatrix}, \tag{12}$$

$$A^{-1} = \begin{pmatrix} A_1^{-1} & 0 & 0 & \dots \\ 0 & A_2^{-1} & 0 & \dots \\ 0 & 0 & \ddots & \\ \vdots & \vdots & & \ddots \end{pmatrix}. \tag{13}$$

Now, according to equations (8), (1), (9), (11) and (10) we have

$$\begin{aligned} \dot{y}(t) = \tilde{\vec{w}}^{\mathrm{t}}\dot{\vec{z}}(t) &= \tilde{\vec{w}}^{\mathrm{t}}A\vec{z}(t) + \tilde{\vec{w}}^{\mathrm{t}}\vec{w}f(t) \\ &= \vec{w}^{\mathrm{t}}\vec{z}(t), \end{aligned} \tag{14}$$

which simply states the fact that the velocity at the observation point is a weighted sum of modal velocities (with the same weighting factors as applied for displacement). Further differentiation of (14) then gives

$$\ddot{y}(t) = \vec{w}^{\mathrm{t}}A\vec{z}(t) + |\ \vec{w}\ |^2\ f(t), \tag{15}$$

which is equivalent to

$$f(t) = -\frac{1}{|\ \vec{w}\ |^2}\vec{w}^{\mathrm{t}}A\vec{z}(t) + \frac{1}{|\ \vec{w}\ |^2}\ddot{y}(t). \tag{16}$$

Substitution of equations (9) and (16) into equation (1) finally gives

$$\begin{aligned} \dot{\vec{z}}(t) &= A\vec{z}(t) - \frac{1}{|\ \vec{w}\ |^2}\vec{w}\vec{w}^{\mathrm{t}}A\vec{z}(t) + \frac{1}{|\ \vec{w}\ |^2}\vec{w}\ddot{y}(t) \\ &= (\mathrm{I} - P_{\vec{w}})A\vec{z}(t) + \frac{1}{|\ \vec{w}\ |^2}\vec{w}\ddot{y}(t), \end{aligned} \tag{17}$$

with I the identity operator and

$$P_{\vec{w}} := \frac{1}{|\ \vec{w}\ |^2}\vec{w}\vec{w}^{\mathrm{t}}. \tag{18}$$

By further defining

$$P := \mathrm{I} - P_{\vec{w}},\ M := PA, \tag{19}$$

equation (17) reads as

$$\dot{\vec{z}}(t) = M\vec{z}(t) + \frac{1}{|\ \vec{w}\ |^2}\vec{w}\ddot{y}(t). \tag{20}$$

which is now seen to be of the same basic structure as equation (1) with the second derivative $\ddot{y}(t)$ playing the role of a "pseudo force" applied with a scaled "weighting vector" $\frac{1}{|\vec{w}|^2}\vec{w}$. More decisive is the new "evolution operator" M which encodes the structure of the scenario of a forced vibration profile and therefore has to be analysed in order to characterise the solution.

Equation (20) already shows that the question at the end of the previous subsection has a positive answer. After what has been said in the introduction the solution is also unique and may be presented in the form (2). In order to specify the solution in practice further analysis is necessary and the structure of the crucial operator M will be seen to defy a conventional straightforward approach.

2.3. Solving the new evolution equation

In order to solve an evolution equation of form (1) with a finite-dimensional state space a transformation of the operator A (in the present case M) to Jordan normal form [5] has to be performed. This is usually done by means of a numerical algorithm of diagonalisation which approximates the eigenvalues of M. (Indeed, the problem of finding the eigenvalues of a matrix can in principle not be solved exactly, by means of a terminating algorithm, compare e.g. [5].) Of course a necessary precondition for numeric algorithms of diagonalisation is that the matrix in question *is* diagonalisable, i.e. that its Jordan normal form is diagonal. In engineering text books this precondition is generally simply assumed to be fullfilled and the case of non-diagonalisable matrices is often claimed to be "not relevant in practice" and thus not considered. It however turns out that the matrix M in equation (20) does *not* fullfill this precondition of diagonalisability. The numeric algorithms usually used for finding eigenvalues and -vectors here generally do not converge "correctly". For solving equation (20) some individual theoretic analysis is therefore necessary.

We start by looking at the operator $P_{\vec{w}}$ (definition (18)) which is seen to be the orthogonal projection onto the subspace $Z_{\vec{w}}$ spanned by the vector $\vec{w}$: $P_{\vec{w}}$ acts on any vector $\vec{x}$ by building its scalar product with $\vec{w}_{\text{norm}} := \frac{1}{|\vec{w}|}\vec{w}$ and subsequent scalar multiplication with the same vector $\vec{w}_{\text{norm}}$. $P_{\vec{w}}$ is accordingly of rank 1 — its range being $Z_{\vec{w}}$ — and its kernel consists of all vectors orthogonal to $\vec{w}$. As any orthogonal projection $P_{\vec{w}}$ is characterised as such by

$$P_{\vec{w}}P_{\vec{w}} = P_{\vec{w}} \tag{21}$$

or equivalently by

$$P_{\vec{w}}\ |_{\text{range}(P_{\vec{w}})} = \mathrm{I}_{\text{range}(P_{\vec{w}})} \tag{22}$$

Again as for every orthogonal projection, $P = \mathrm{I} - P_{\vec{w}}$ is also an orthogonal projection with

$$\text{range}(P) = \text{kern}(P_{\vec{w}}) \text{ and } \text{kern}(P) = \text{range}(P_{\vec{w}}) = Z_{\vec{w}}. \tag{23}$$

Since the operator A defined by (4) and (5) is invertible we have

$$\text{kern}(M) = A^{-1}\text{kern}(P) = A^{-1}Z_{\vec{w}}. \tag{24}$$

The kernel of M is thus of dimension 1, spanned by

$$\vec{v}^{(1)} := A^{-1}\vec{w}, \tag{25}$$

which is just another way of saying that 0 is an eigenvalue of M and $\vec{v}^{(1)}$ being the only according eigenvector (up to scaling). There might however be further *generalised* eigenvectors to the eigenvalue 0 which is the case if there exists any vector $\vec{v}^{(2)}$ such that $\vec{v}^{(1)} = M\vec{v}^{(2)}$. Taking into account equations (11) and (10) we compute $\vec{w}^{\rm t}\vec{v}^{(1)} = \tilde{\vec{w}}^{\rm t}AA^{-1}\vec{w} = \tilde{\vec{w}}^{\rm t}\vec{w} = 0$. So, $\vec{v}^{(1)}$ is orthogonal to $\vec{w}$ and, after what has been said above, in the kernel of $P_{\vec{w}}$, thus in the range of P and therefore in the range of $M(=PA)$. Due to the projection properties of P we further have $P\vec{v}^{(1)} = \vec{v}^{(1)}$. After these arguments a $\vec{v}^{(2)}$ as above must exist ("$P_{\vec{w}}$ is in the range of M") and can be chosen as

$$\vec{v}^{(2)} := A^{-1}\vec{v}^{(1)} = A^{-2}\vec{w}. \qquad (26)$$

We confirm: $M\vec{v}^{(2)} = PAA^{-1}\vec{v}^{(1)} = P\vec{v}^{(1)} = \vec{v}^{(1)}$. These arguments already show that the operator M can not be diagonalised: its Jordan normal form must at least contain one non-diagonal Jordan block, the one to the eigenvalue 0.

At this point it is useful to compute the explicit components of the vectors $\vec{v}^{(1)}$ and $\vec{v}^{(2)}$ which after relations (25) and (26) are

$$\vec{v}^{(1)} = -(w_1/k_1, 0, w_2/k_2, 0, \ldots)^{\rm t} \qquad (27)$$

and

$$\vec{v}^{(2)} = (c_1 w_1/k_1^2, -w_1/k_1, c_2 w_2/k_2^2, -w_2/k_2, \ldots)^{\rm t}. \qquad (28)$$

In order to check if M has any further generalised eigenvectors to the eigenvalue 0 we compute $\vec{w}^{\rm t}\vec{v}^{(2)} = \tilde{\vec{w}}^{\rm t}AA^{-2}\vec{w} = \tilde{\vec{w}}^{\rm t}A^{-1}\vec{w} = -\sum w_n^2/k_n$. This value is non-zero if at least one w_n is (which we clearly assume), therefore $\vec{v}^{(2)}$ is *not* orthogonal to $\vec{w}$ and thus not in the range of P, i.e. of M. So, no further generalised eigenvectors to the eigenvalue 0 exist and the according Jordan block is of size 2×2.

Full transformation of the matrix M into Jordan normal form is achieved by proceeding with a process of deflation which is shortly summarized in the following. The vectors $\vec{v}^{(1)}$ and $\vec{v}^{(2)}$ are completed to a basis of Z by the unity vectors $\vec{e}^{(3)} = (0, 0, 1, 0, 0, \ldots)^{\rm t}$, $\vec{e}^{(4)} = (0, 0, 0, 1, 0, 0, \ldots)$, ... In order to see that this set of vectors really forms a basis the determinant of the matrix

$$V_1 := \left(\vec{v}^{(1)}, \vec{v}^{(2)}, \vec{e}^{(3)}, \vec{e}^{(4)}, \ldots\right), \qquad (29)$$

must be computed which, using the component expressions (25) and (26), turns out to be $\mid V_1 \mid = w_1^2/k_1^2$, and without loss of generality we assume that $w_1 \neq 0$. After the properties of the generalised eigenvectors $\vec{v}^{(1)}$ and $\vec{v}^{(2)}$ above we have

$$MV_1 = V_1 \begin{pmatrix} 0 & 1 & \vec{a}^{\rm t} \\ 0 & 0 & \vec{b}^{\rm t} \\ \vdots & \vdots & N \end{pmatrix} =: V_1 M_1, \qquad (30)$$

with some row vectors $\vec{a}^{\rm t}$ and $\vec{b}^{\rm t}$ and a "deflated" matrix N which may be computed according to the equivalent equation

$$M_1 = V_1^{-1} M V_1. \qquad (31)$$

(The inverse of the matrix V_1 may be easily computed even "by hand".) For the matrix N we may now assume that it *is* diagonalisable and apply a numeric algorithm of choice to find its transformation to diagonal form: any further non-diagonal Jordan block would — in contrast to the one belonging to the eigenvalue 0 which is an unavoidable result of the nature of the posed problem itself — be a sign of "unusual symmetry" in the parameters k_n, c_n, w_n.

Of course in any practical implementation of the receipt presented here the result of such a numeric algorithm for diagonalisation should be cross-checked (which has been done for the example presented further below and others to no unpleasant surprises). The result of a successful such algorithm will be transformation matrices satisfying $NU_N = U_N D$ with D diagonal, containing the eigenvalues $d_3, d_4, \ldots$ of N and U_N invertible holding the according eigenvectors (of N) $\vec{u}_N^{(3)}, \vec{u}_N^{(4)}, \ldots$ as its columns. It is easily checked (by looking at the structure of the characteristic polynomials) that all eigenvalues d_n of N are also eigenvalues of M_1 (and thus due to the similarity transformation (30) also of M which however will again become apparent later). For each eigenvector $\vec{u}_N^{(n)}$ of N an eigenvector $\vec{u}^{(n)}$ of M_1 can be constructed in the following way:

$$\vec{u}^{(n)} := \begin{pmatrix} u_1^{(n)} \\ u_2^{(n)} \\ \vec{u}_N^{(n)} \end{pmatrix}, \qquad (32)$$

$$u_2^{(n)} := \frac{1}{d_n}\vec{b}^{\rm t}\vec{u}_N^{(n)}, \; u_1^{(n)} := \frac{1}{d_n}(\vec{a}^{\rm t}\vec{u}_N^{(n)} + u_2^{(n)})$$

(These definitions follow canonically from the according equations in component form.) The set of eigenvectors $\vec{u}_N^{(n)}$ of N is linearly independent (U_N is invertible) and so is the set $\vec{u}^{(n)}$ as well as the set $\vec{e}^{(1)}, \vec{e}^{(2)}, \vec{u}^{(3)}, \vec{u}^{(4)}, \ldots$. (Again this is easy to confirm.) With the latter we have therefore found a basis of Z consisting of generalised eigenvectors of M_1 which we combine into an (invertible) matrix

$$U := \left(\vec{e}^{(1)}\vec{e}^{(2)}\vec{u}^{(3)}\vec{u}^{(4)}\ldots\right). \qquad (33)$$

The matrix U now delivers the transformation of M_1 to Jordan normal form:

$$M_1 U = U \begin{pmatrix} 0 & 1 & 0\ldots \\ 0 & 0 & 0\ldots \\ \vdots & \vdots & D \end{pmatrix} =: UJ \qquad (34)$$

Combining equations (34) and (30) and defining

$$V := V_1 U \qquad (35)$$

we finally find

$$MV = VJ, \qquad (36)$$

the transformation of M to Jordan normal form.

2.4. Remarks and receipt summary

Some important aspects of the just presented method shall be shortly summarised. Equation (20) proves that the problem posed at the end of subsection 2.2 has *one, unique* solution. The nature of this solution is characterised by the transformation of the operator M to Jordan normal form which has been shown to contain one non-diagonal Jordan block to the eigenvalue 0. Such a Jordan block with 0 values on its diagonal accords to a "free–mass behaviour" which reflects the fact that the objects is forced to follow the vibration profile $y(t)$ at one point. This non-diagonal block can thus be seen as representing a reduction (with respect to the free modal object) of the degrees of freedom of the system. The remaining eigenvalues of M with regular eigenspaces accord to the modal frequencies of the object under the additional constraint of being fixed at the observation point. Of course the according modal shapes may also be quantified from the data in J and V. The presented method thus also forms a technique of deriving modal data of the system build from a known modal object (existing or theoretical) by keeping one point fixed. The information of the resulting shapes of modes and the new equilibrium is contained in the matrix V; details are not elaborated in this contribution.

Summing up, the presented method consists in the following schedule:

- Build the matrices A, $P_{\vec{w}}$ and P from the given modal data.
- Compute M, $\vec{v}^{(1)}$ and $\vec{v}^{(2)}$ and V_1^{-1} according to equations (18), (19), (27), (28) and (29). .
- Build M_1 and extract N, $\vec{a}$ and $\vec{b}$ according to equations (31) and (30).
- Diagonalise N (with any suitable numeric algorithm) and build U according to equations (32) and (33).
- Build J and compute V according to (34) and (35).

3. EXAMPLE

To show the practicability of the method just developed a simple example has been implemented, modelling a stiff string being "hammered" down by an infinite mass against a fixed board. While this scenario might of course be modelled most efficiently by means of a digital waveguide it is a good candidate to check the potentials of the new method since the well-studied and intuitive behaviour of vibrating strings forms a good reference point. In contrast to waveguide simulation the modal description of the stiff string avoids problems of numeric dispersion. All used modal data is gained from the exact theoretic solution of the differential equation of the stiff string, details are omitted here since they can be found in standard text books of acoustics (see e.g. [1]). The simulation has been performed by means of a suitable discrete–time transition matrix, thus all computation is of theoretic exactness up to the limits of the computing architecture and the numeric diagonalisation algorithm (Matlab's "Eig" routine).

Figure 1 shows stages of the simulated string behaviour. The small ripples near the initial contact result from the finite number of modes used in the computation, 100 in the present case.

4. CONCLUSIONS

A method has been developed for handling a specific inverse problem, an object in modal description being forced to follow a given vibration profile at one point. The presented method also forms a technique for computing modal data for a system consisting of a known modal object with the additional constraint of being held fixed at one point. Furthermore it allows to determine the equilibrium state of the system with a forced, constant displacement.

The new technique has been applied at the (comparatively) simple example of a stiff string being hammered against a fixed board to demonstrate its practicability. It shall be applied in the future to more complex settings such as beams, membranes and plates. The presented technique will be very usefull for analysis of measurements — recovering force profiles from measured movement behaviour — and for the simulation of certain mechanisms of musical sound production such as plucked and struck strings and bars and playing techniques of percussion instruments (conga, bongo...) which involve playing gestures of "pushing on the membrane".

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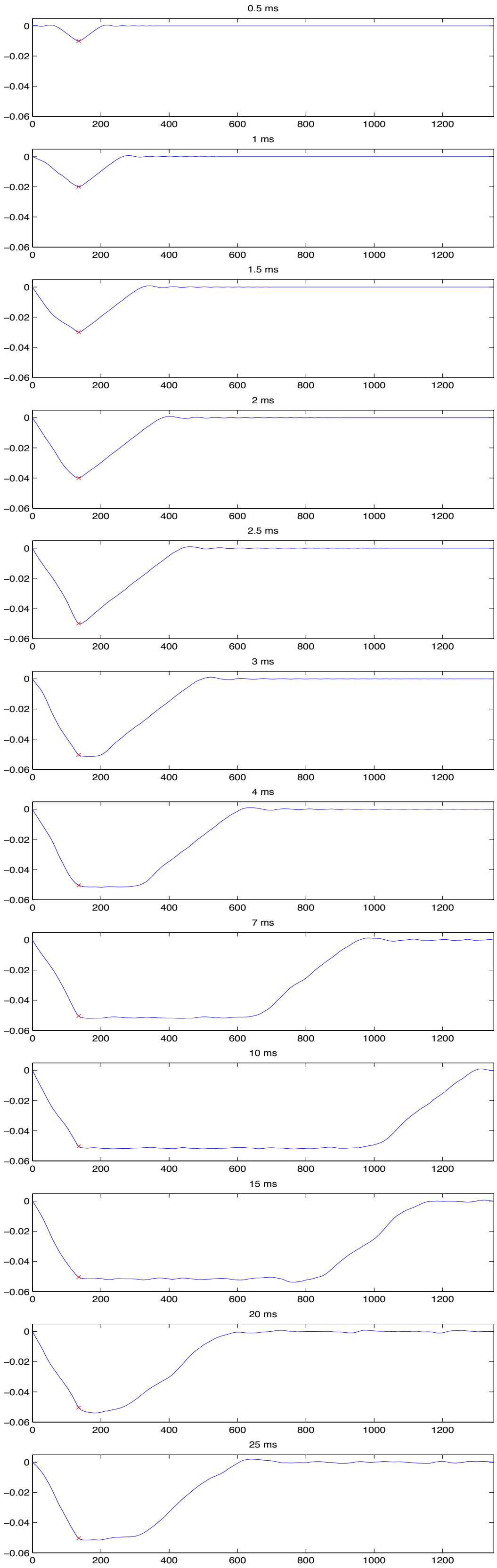

Figure 1: *Snapshots of a string hammered by an "infinite" mass against a fixed soundboard.*